MATHEMATICAL FRONTIERS

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Board on
Mathematical Sciences & Analytics

MATHEMATICAL FRONTIERS

2018 Monthly Webinar Series, 2-3pm ET

February 13: ***Recording posted***
Mathematics of the Electric Grid

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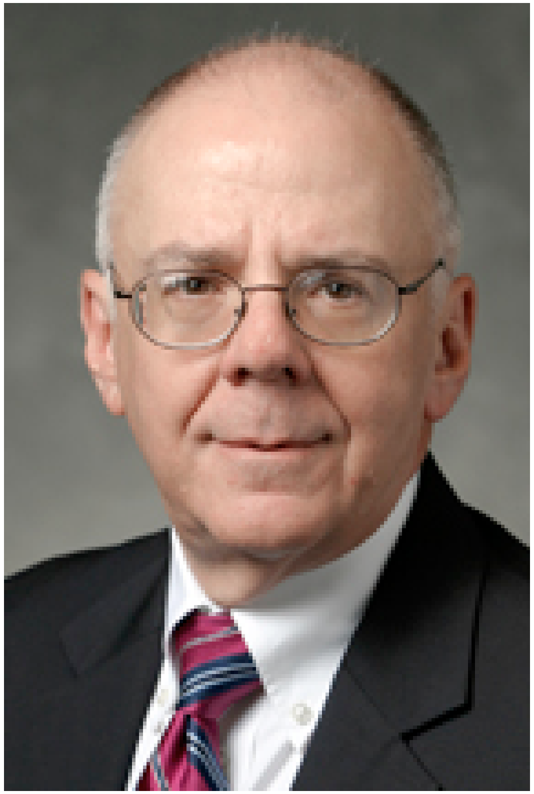
MATHEMATICAL FRONTIERS

Topology



Genevieve Walsh,
Tufts University

Jeffrey F. Brock
Yale University

Joseph Langsam,
University of Maryland

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MATHEMATICAL FRONTIERS
Topology

Genevieve Walsh,
Tufts University

Chaire Jean-Morlet (CIRM) and Professor of Mathematics in the Department of Mathematics

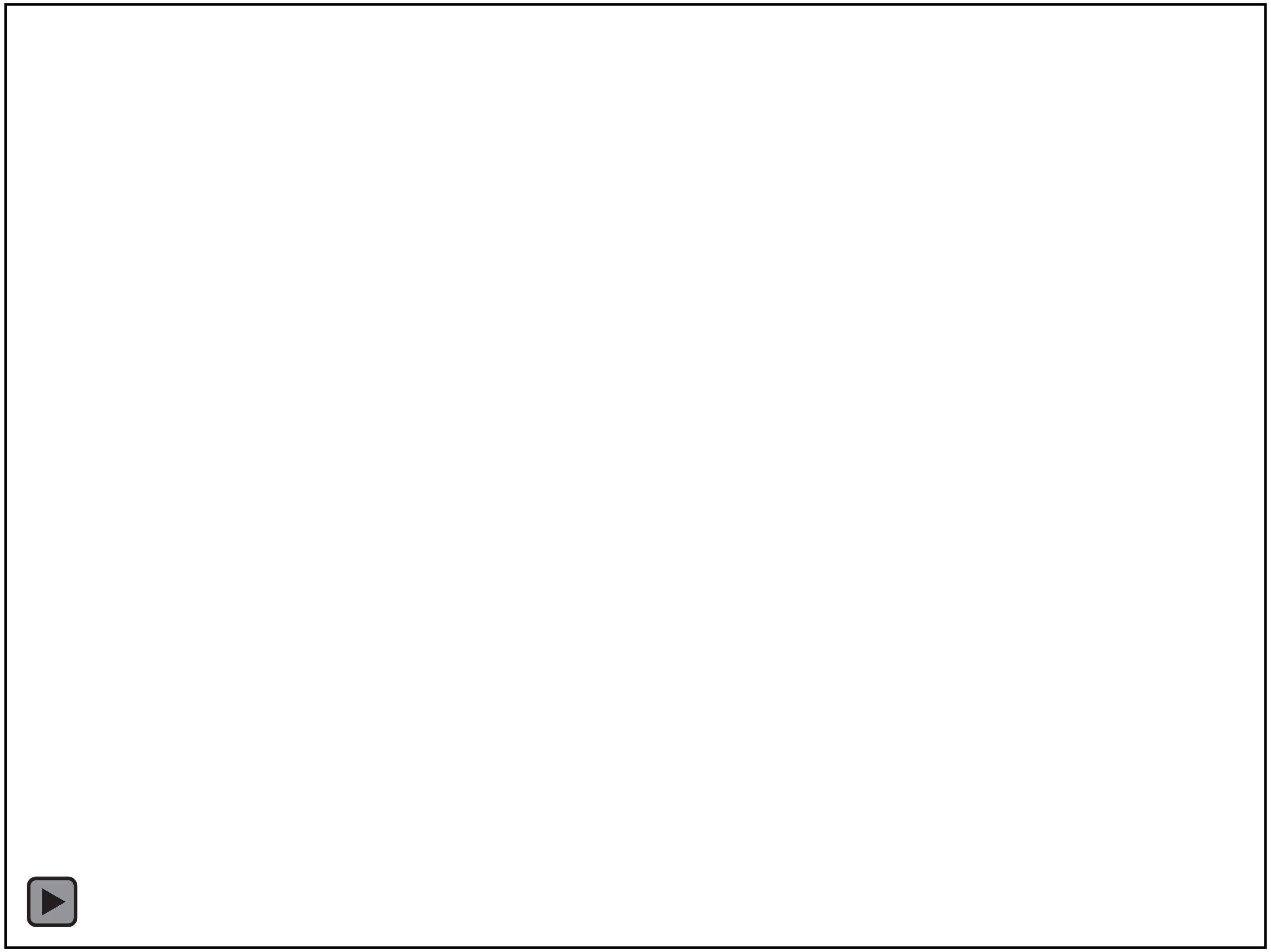
What is topology?

General idea

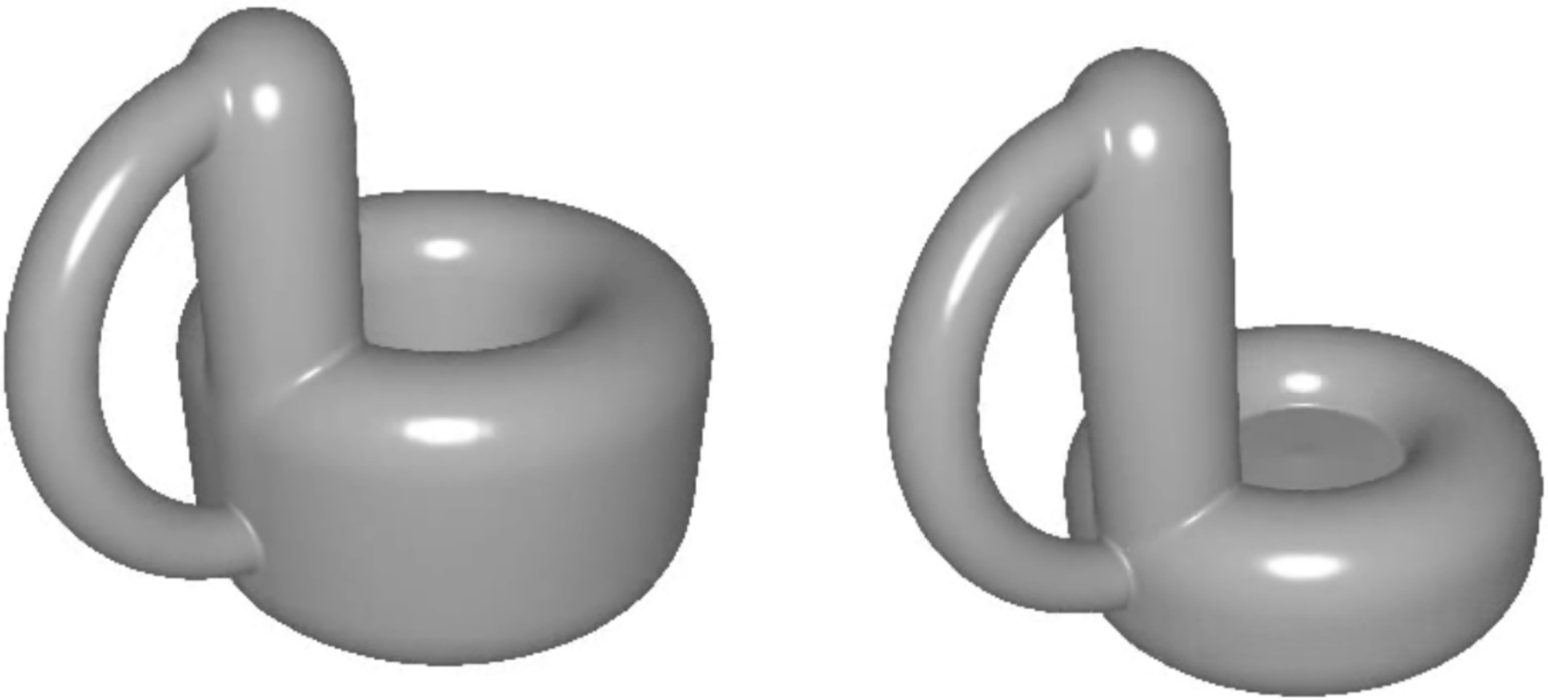
Topology is the study of properties that persist under continuous changes.

How can we make this mathematically precise?

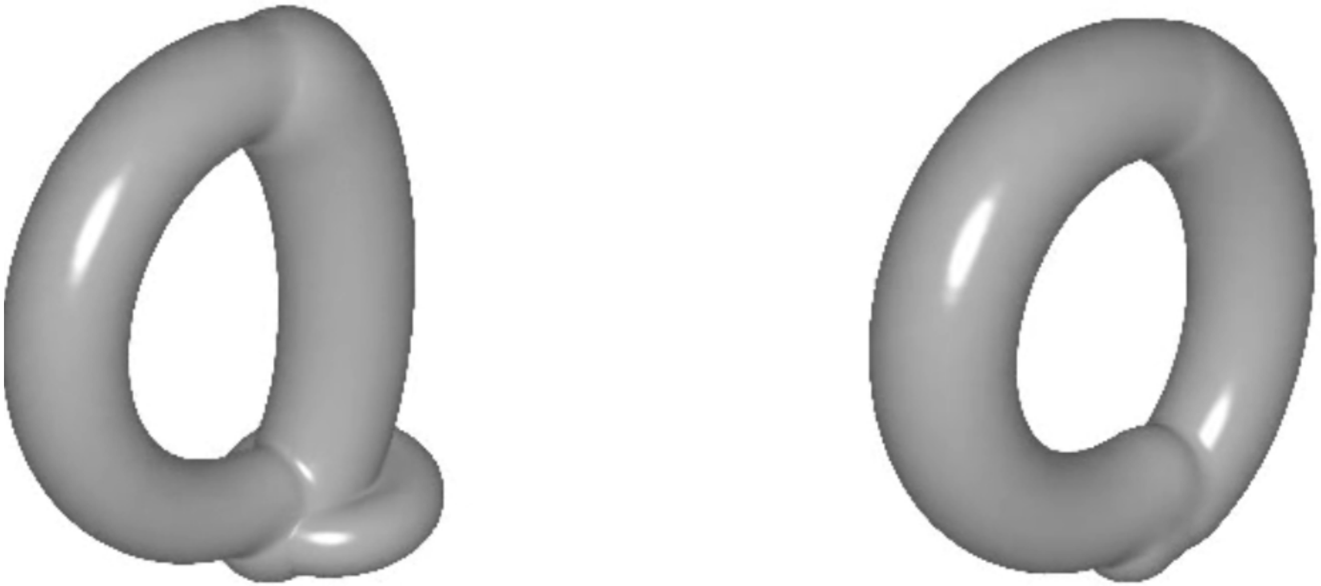
Coffee cup vs donut



Coffee cup…



donut



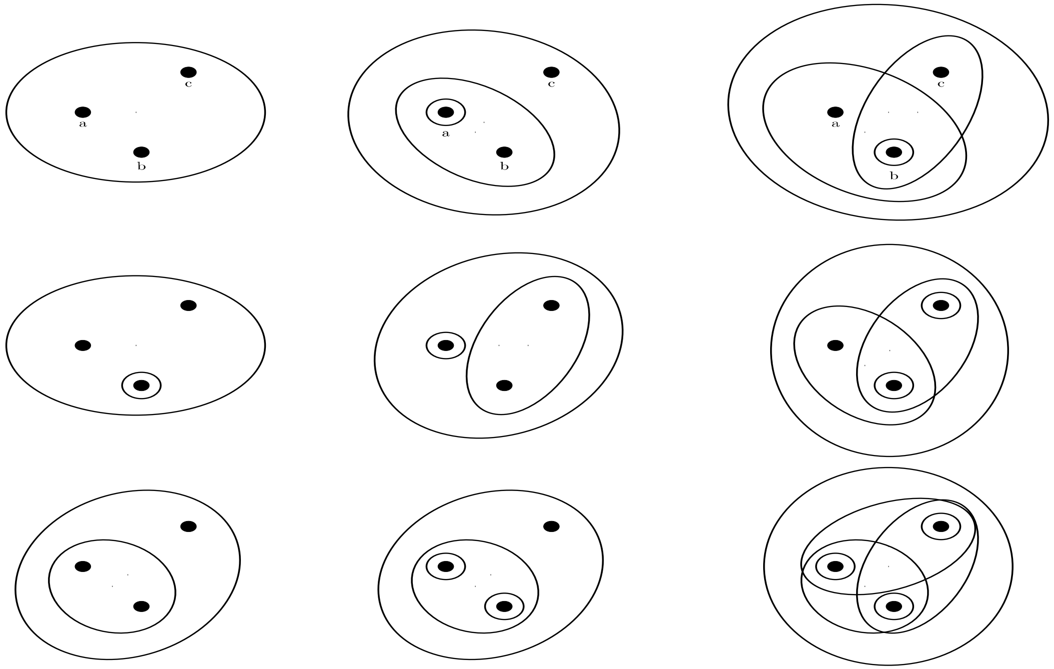
(Thanks Jim Fowler)

Open sets

Given a set X: a TOPOLOGY on X is a collection of subsets of X (the open subsets) such that:

- X is open and $\emptyset$ is open
- the intersection of any two sets is open
- the union of any collection of open sets is open

Examples: three point set



Standard topology on R^n

TOPOLOGY on R^n:

$$\{S: S \; is \; a \; union \; of \; open \; balls\} \cup \emptyset$$

Open balls are sets of the form:

$$B_\varepsilon(x) = \{p \in R^n: d(x,p) < \varepsilon\}$$

Other Topologies on R^n

Discrete Topology on R^n:

$$Every\ subset\ is\ open$$

Power-point topology τ_p on R^n:

$$\{S \subseteq R^n : p \in S\} \cup \emptyset$$

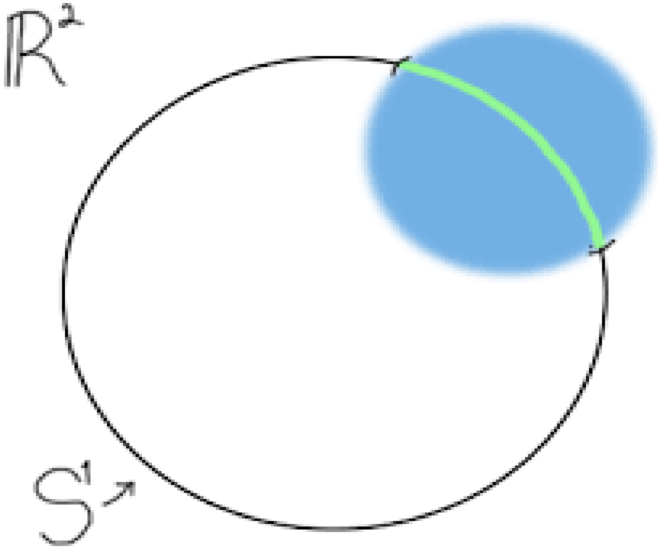
Subspace topology

Given a topological space (X, τ) and a subset $Y \subset X$, the subspace topology on Y, (Y, τ_X) is the collection of sets

$$\tau_X = \{U \cap Y : U \in \tau\}$$

Ex: S^1

The open sets are intersections of open sets in R^2 with S^1



Homeomorphisms

A continuous map between two topological spaces (X, τ) and (Y, ρ) is a function

$$f : X \rightarrow Y$$

Where the pre-image of each open set in (Y, ρ) is open in (X, τ).

A **homeomorphism** is a continuous map with a continuous inverse.

Properties

What properties are preserved under homeomorphisms? Or continuous families of homeomorphisms?

- Connected
- Simply connected (trivial fundamental group)

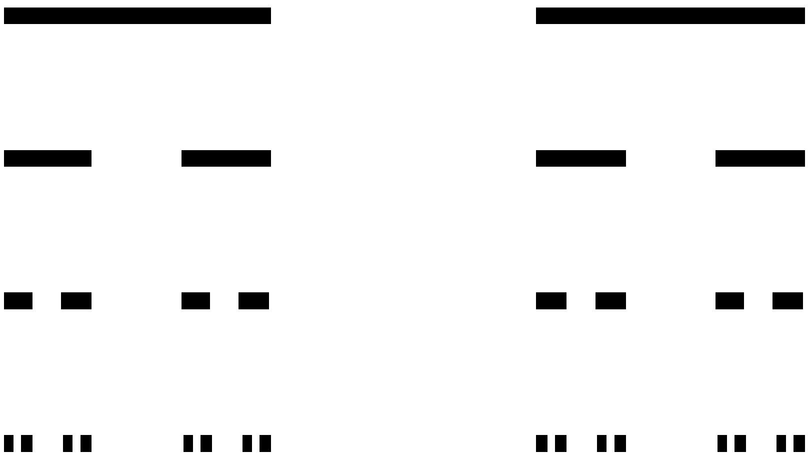
Connected vs non-connected

A topological space is connected if it cannot be written as the union of two clopen sets.

(clopen = open and closed!)

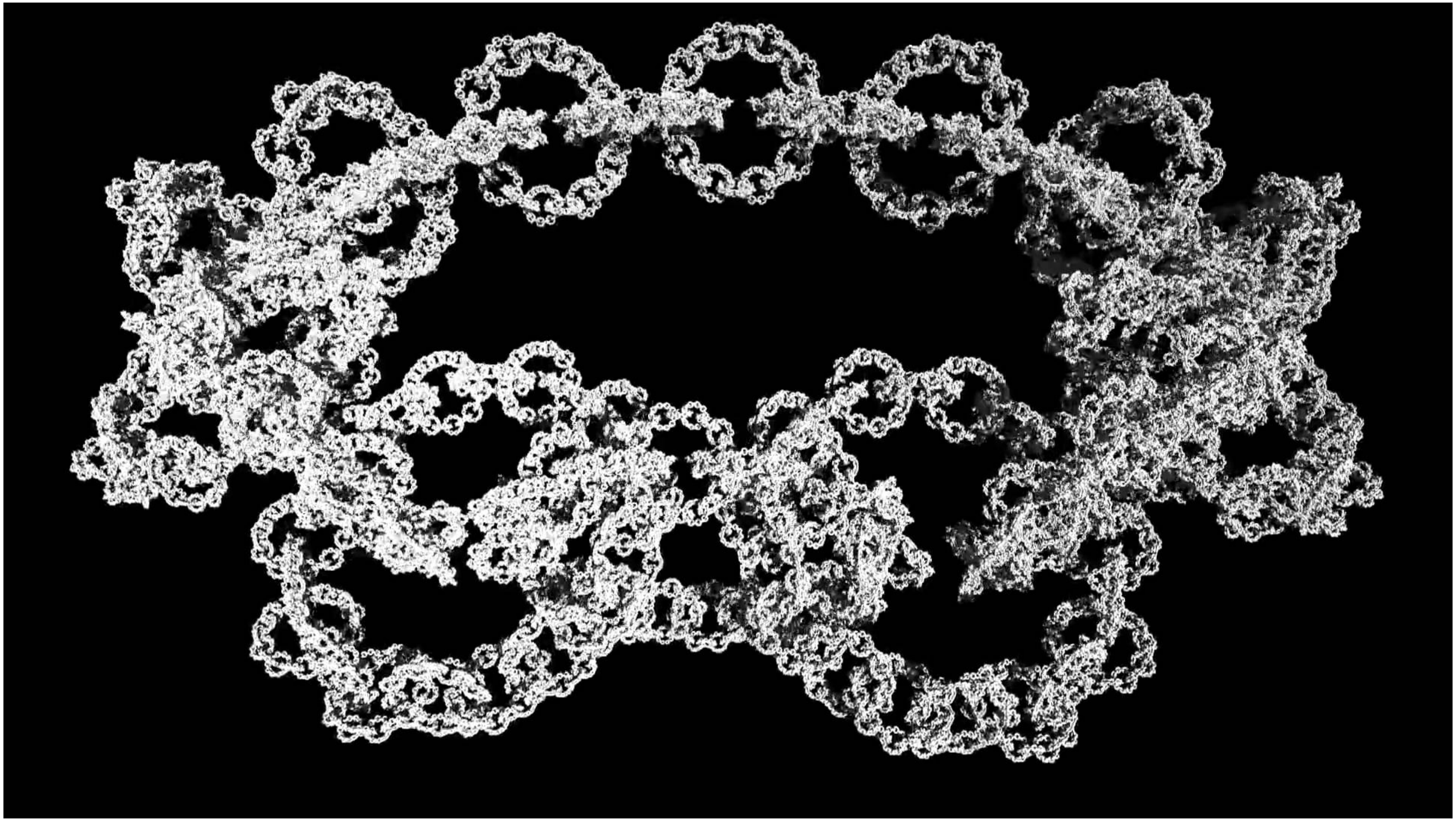
Ex: The Cantor set is not connected (use the subspace topology!)

The Cantor Set



The limit is a Cantor set …

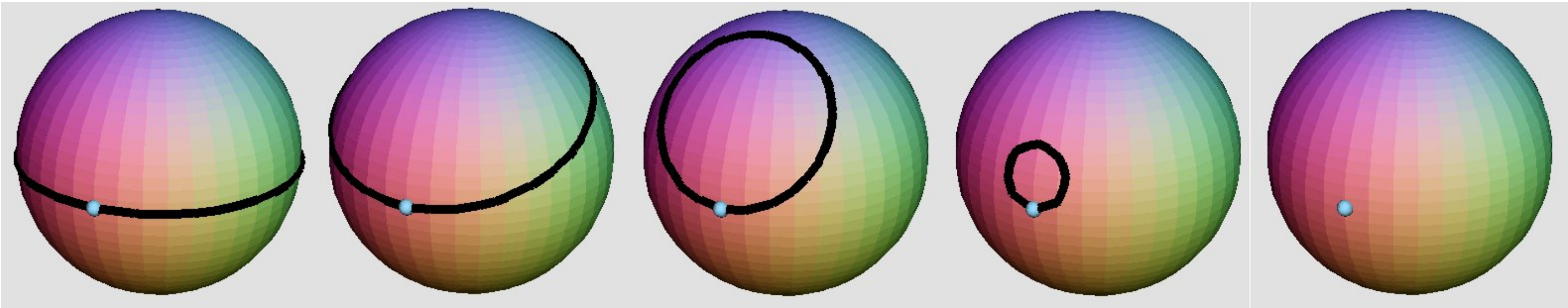
Another picture of the cantor set



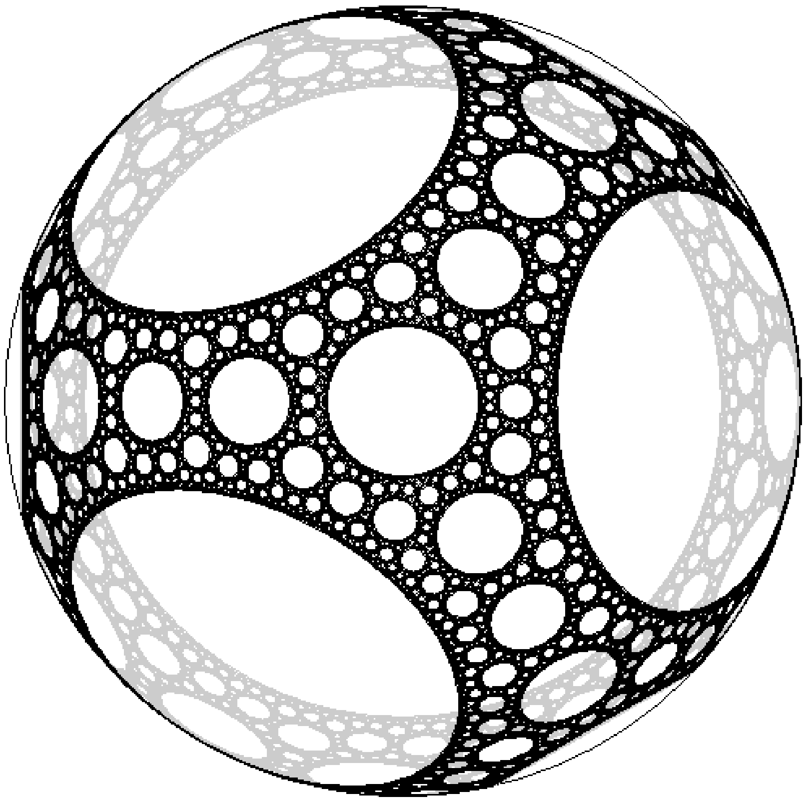
Simply connected

- A path-connected space is simply connected if every loop can be shrunk to a point.
- Formally: any continuous map $f: S^1 \to X$ can be extended to $F: D^2 \to X$

The sphere is simply connected



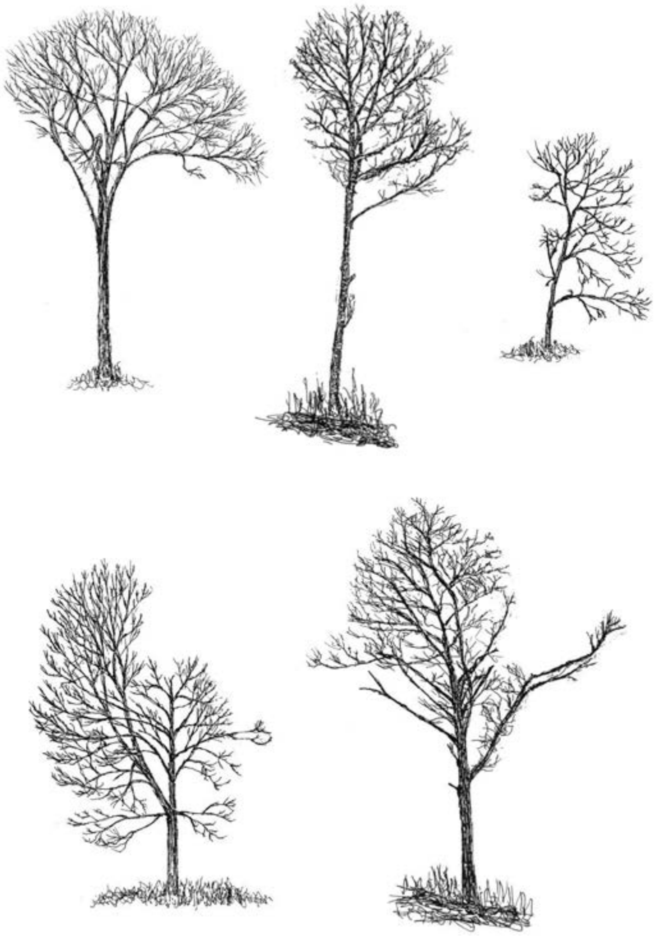
The Sierpinski carpet is not simply connected



Cut points

A cut point of a topological space (X, τ) is a point so that $X \setminus p$ is not connected.

Cut points detect tree-like nature



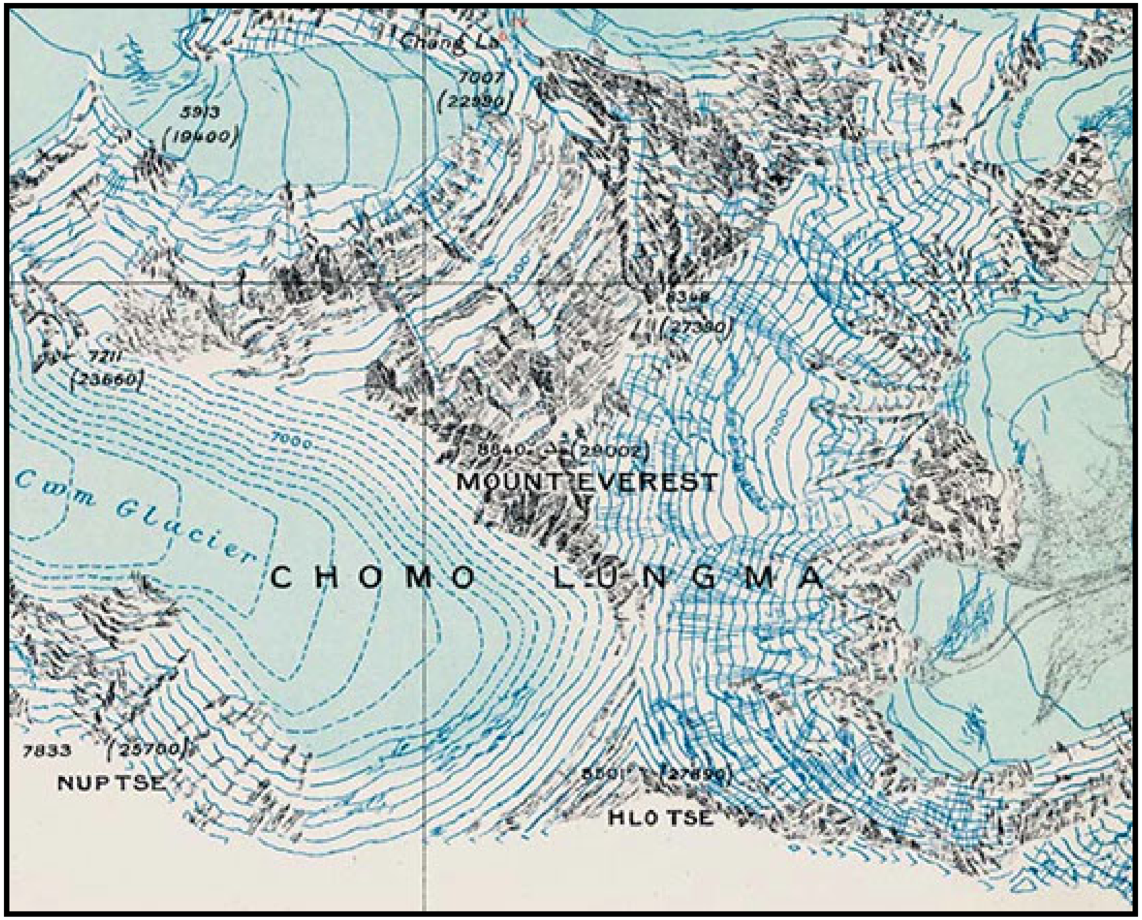
MATHEMATICAL FRONTIERS
Topology

Jeffrey F. Brock,
Yale University

Incoming Dean of Science
and Professor of Mathematics
in the Department of
Mathematics

Applied Topology:
from data science to neuroscience to social networks

Topology



Chang La
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(22990)
5913
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7211
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7000
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MOUNT EVEREST
Cwm Glacier
CHOMO LUNGMA
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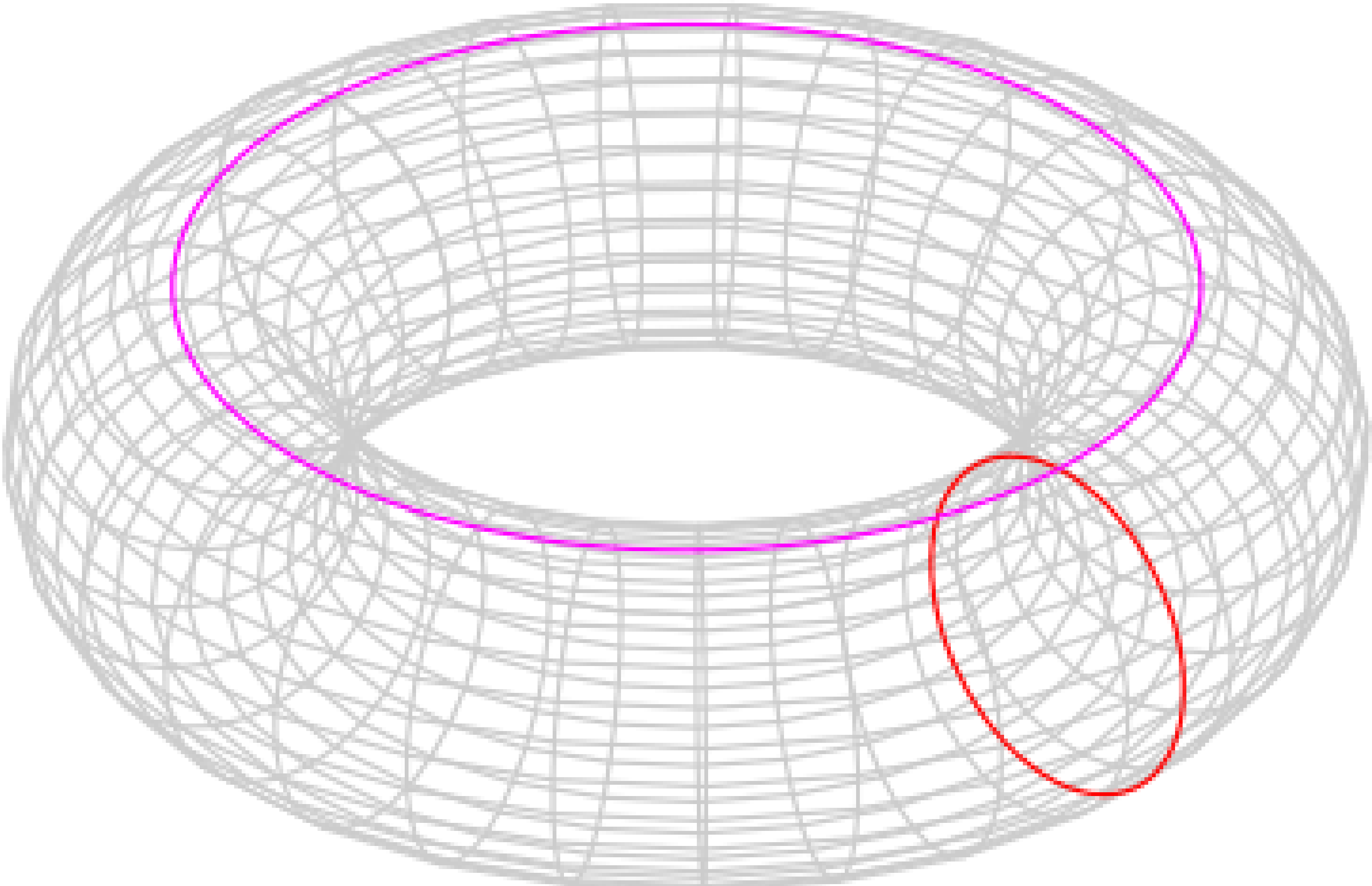
(25700)
NUPTSE
8501
(27890)
HLO TSE

Topology

Topology

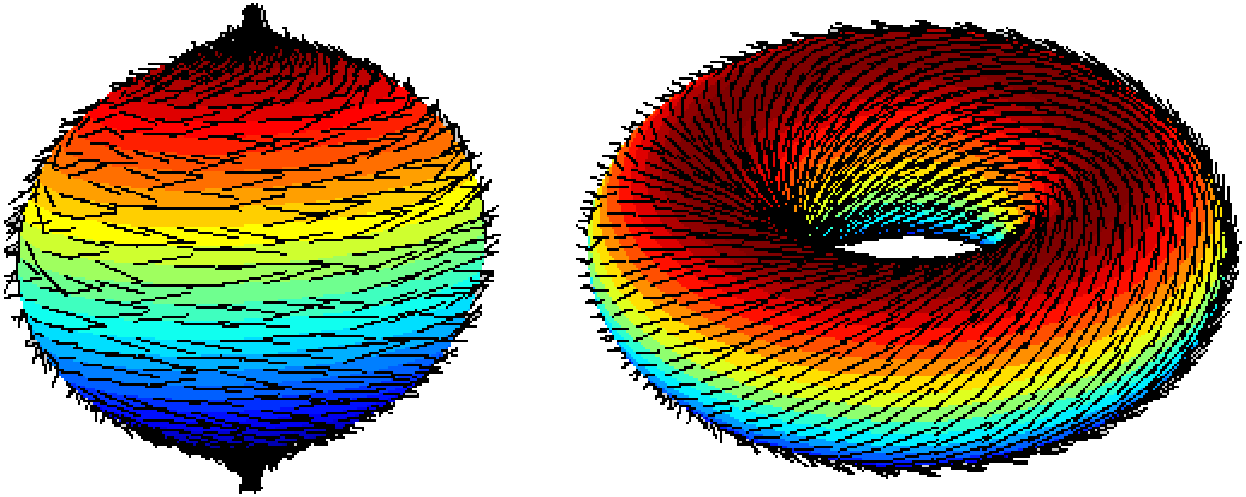


Topology

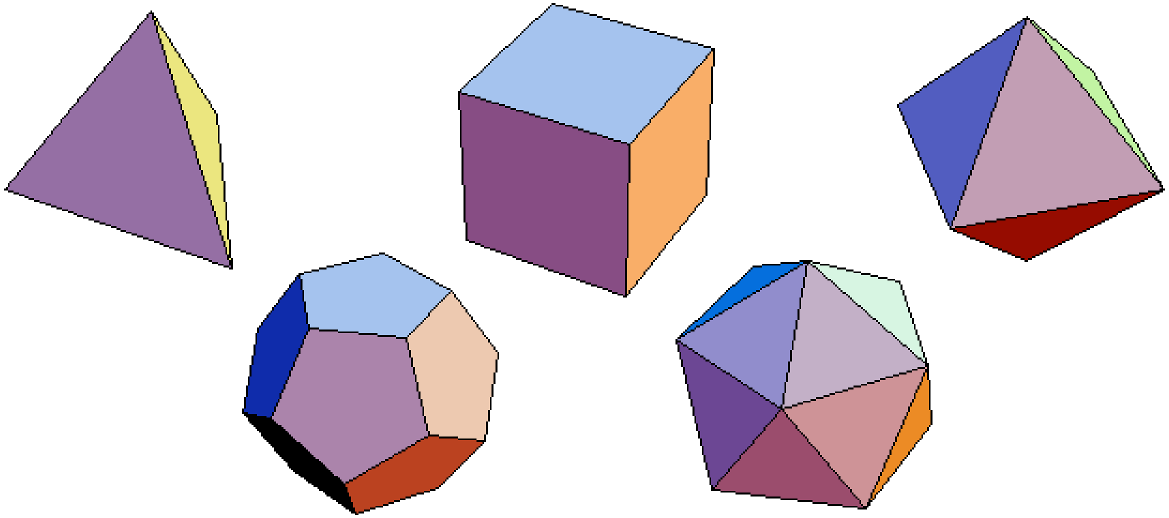
- **Sample Theorem**: there is no non-vanishing continuous tangent vector field on the sphere.

You cannot comb the hair of a sphere: there will always be a bald spot

Topology



Classical Polyhedra

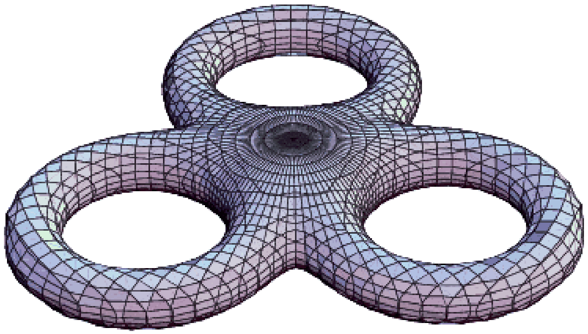


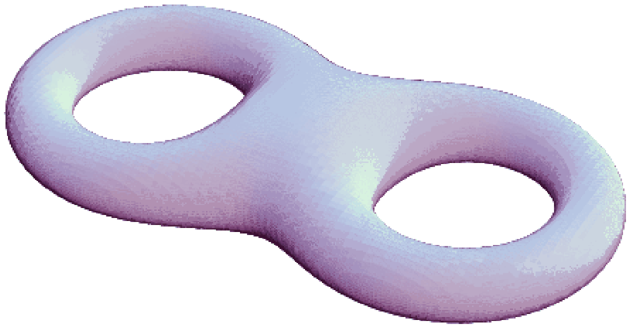
Count the Vertices, the Edges, and the Faces

Is there a pattern? $V - E + F$?

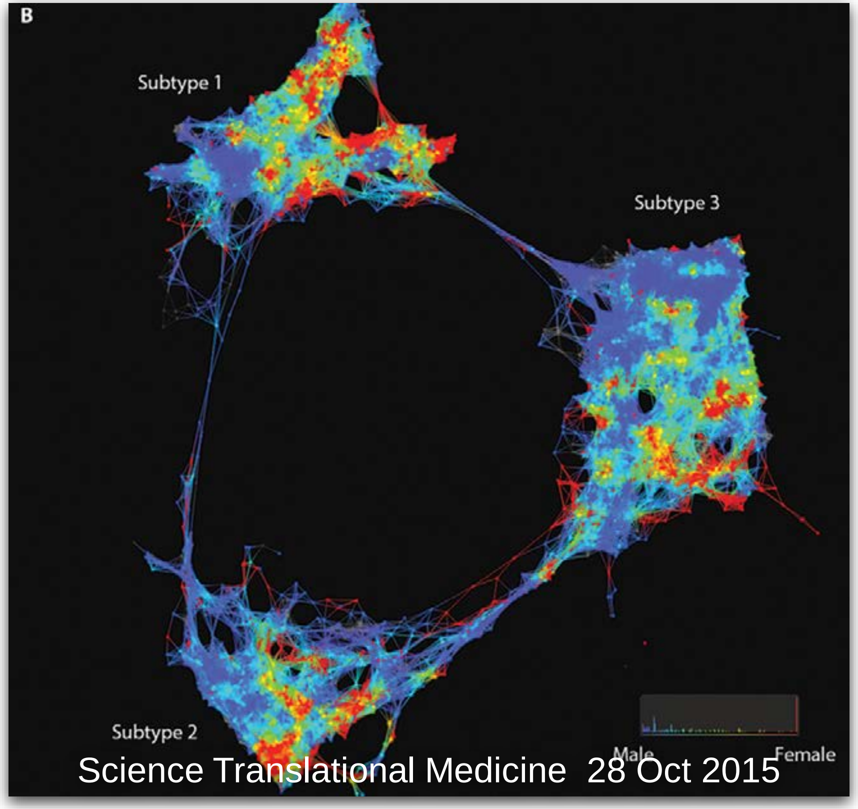
Surfaces

- Surfaces come in many shapes and sizes
- The “number of holes”or g classifies the surface ***topologically***.
- $V - E + F = 2 - 2g$ - the ***Euler number*** does as well!



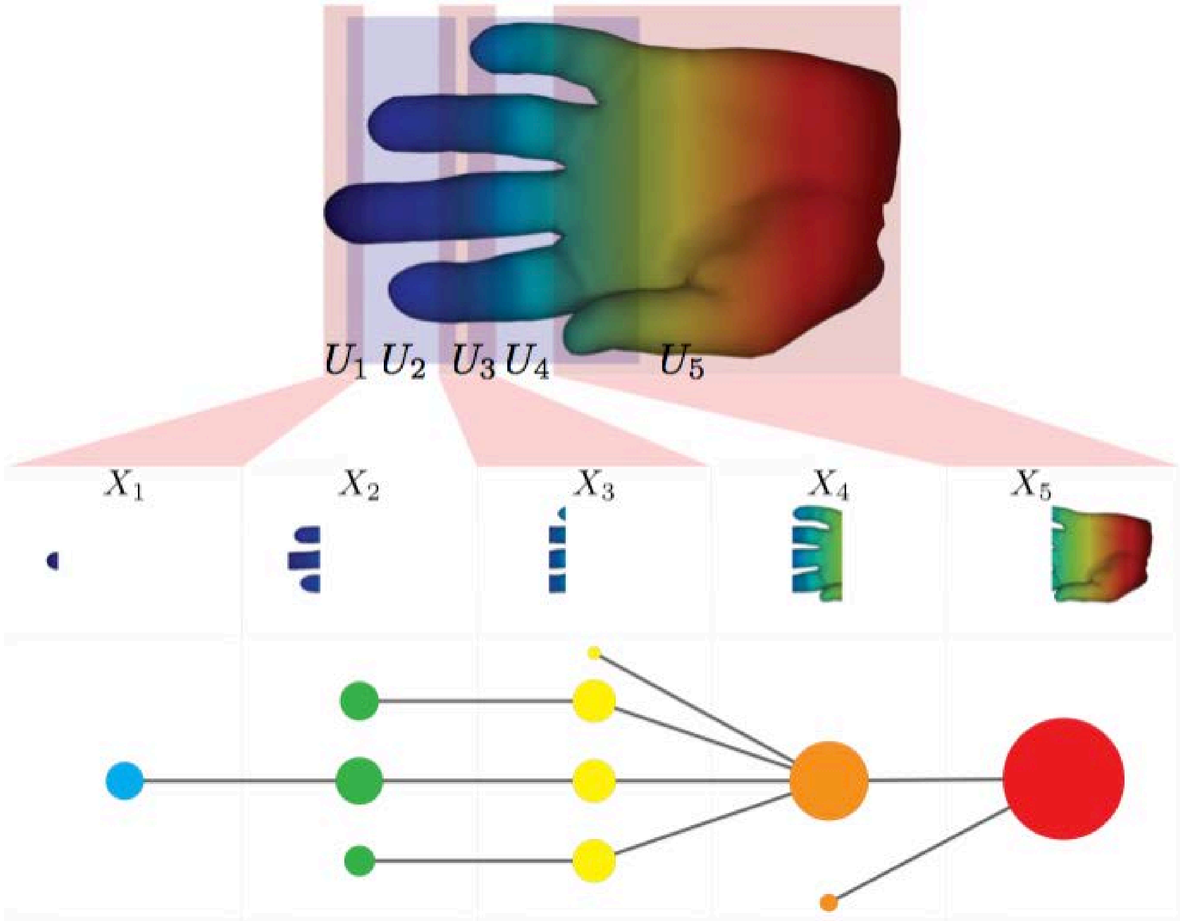


Applied Topology

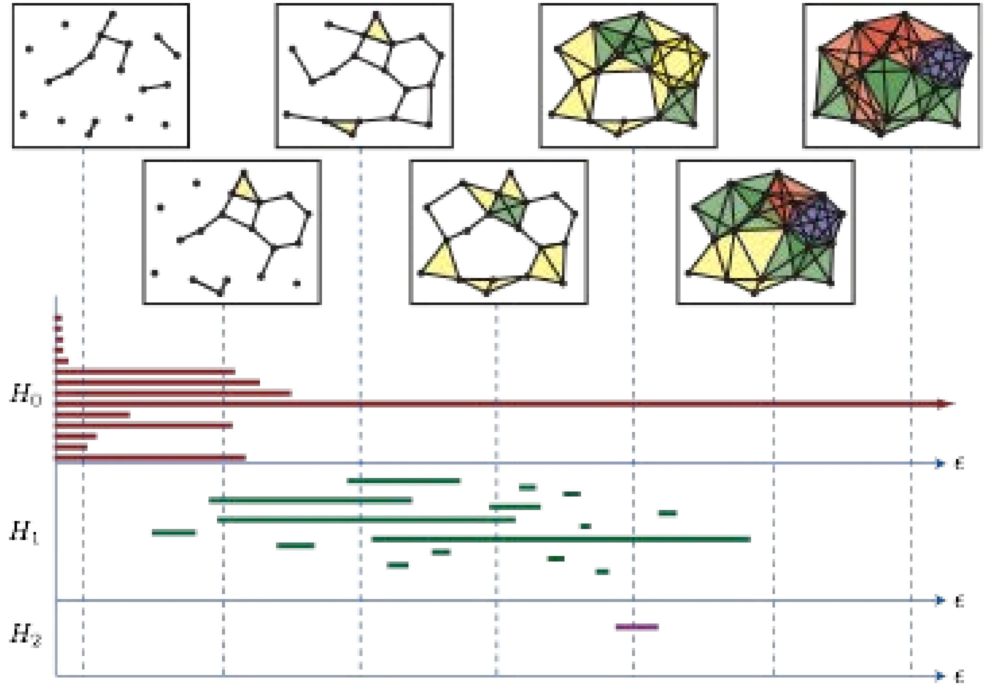


Li Li1, Wei-Yi Cheng1, Benjamin S. Glicksberg1, Omri Gottesman2, Ronald Tamler3, Rong Chen1, Erwin P. Bottinger2 and Joel T. Dudley

Topological Signatures

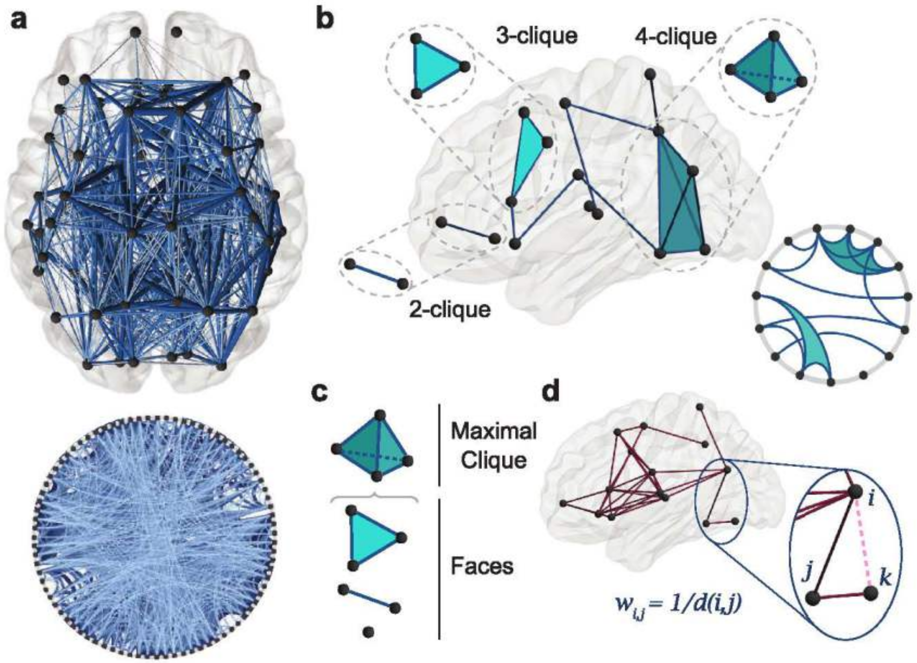


Barcodes



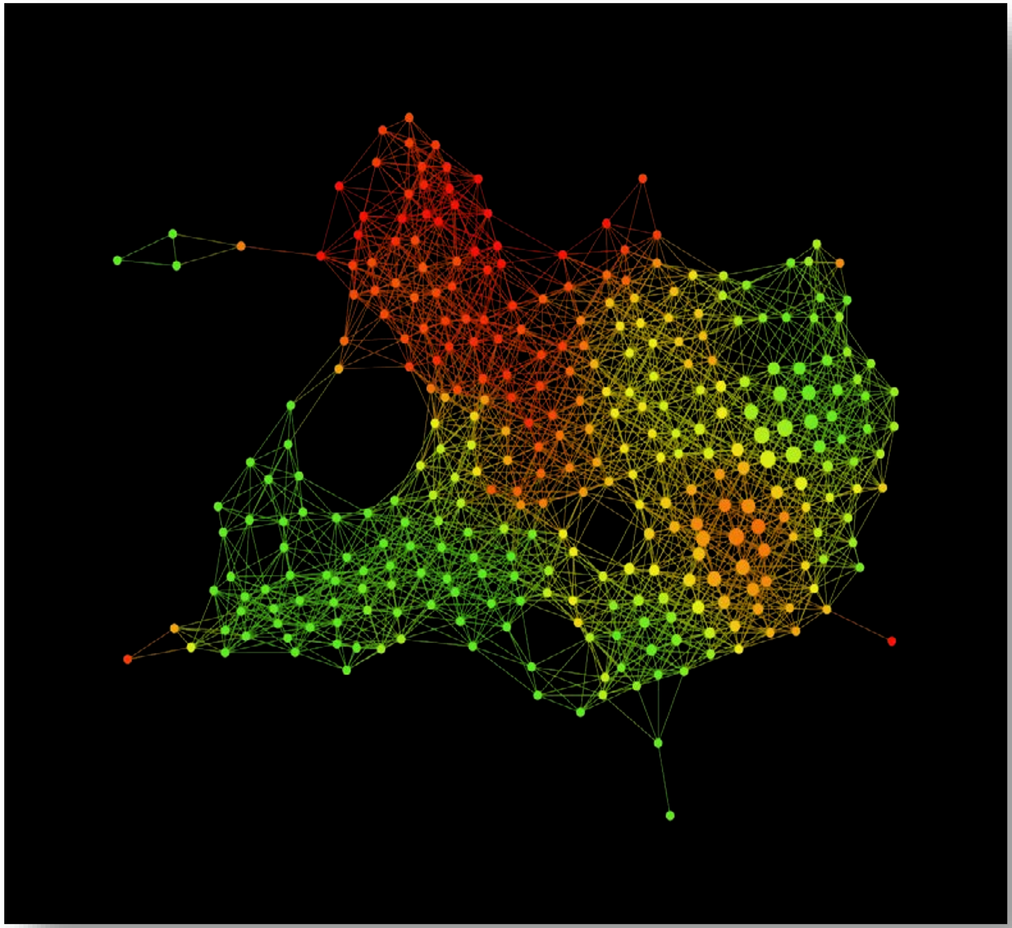
R. Ghrist - *Barcodes*

Cliques in the Connectome



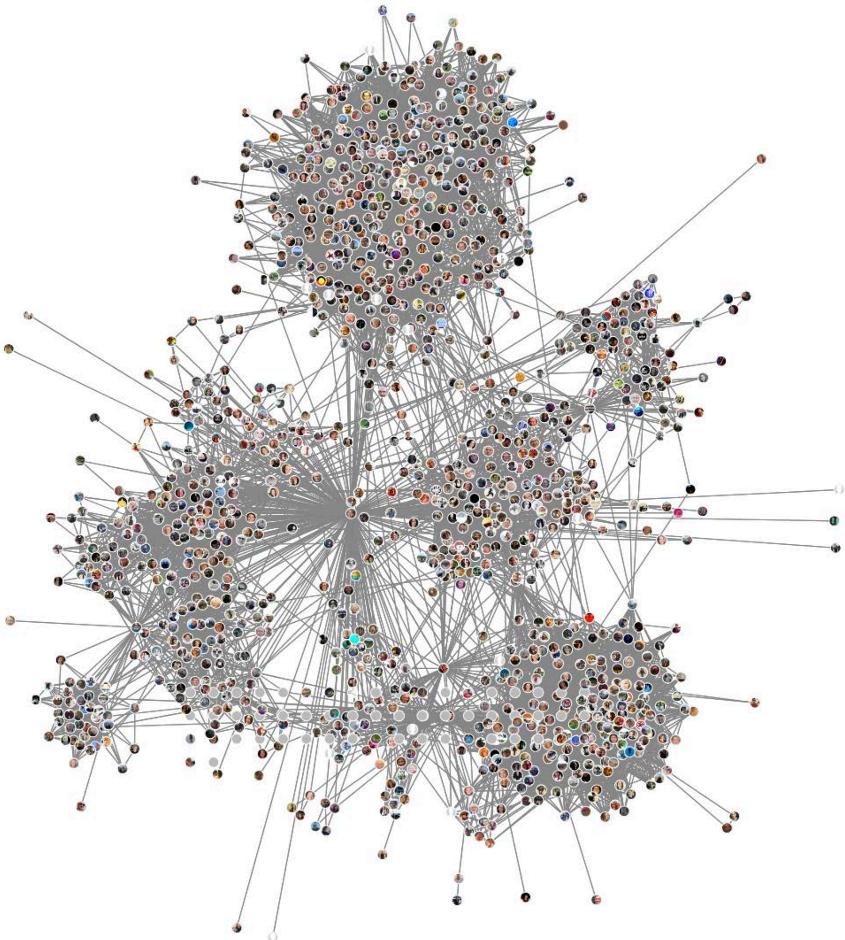
Ann Sizemore, Chad Giusti, Richard F. Betzel, Danielle S. Bassett - [arXiv:1608.03520]

Radiology



Krishna N. Keshavamurthy ; Owen P. Leary ; Lisa H. Merck ; Benjamin Kimia ; Scott Collins ; David W. Wright ; Jason W. Allen ; Jeffrey F. Brock ; Derek Merck

Social Networks



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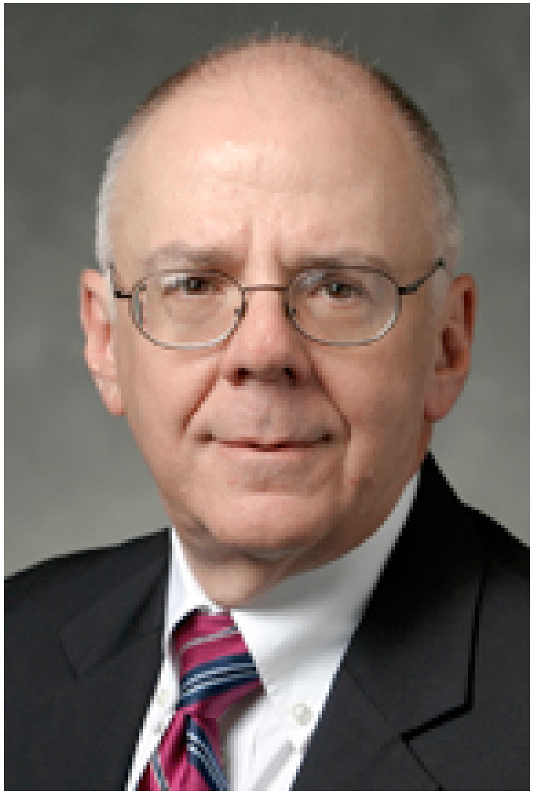

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