Localization of Waves

Svitlana Mayboroda

September 2018

The results presented in this talk are a part of a newly established Simons collaboration "Localization of Waves" including the PIs

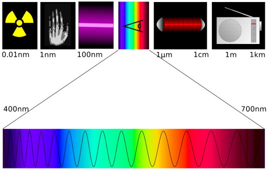
- D. Arnold, UMN (applied math)
- A. Aspect, Institut d'Optique (cold atoms)
- G. David, Université Paris-Sud (harmonic analysis, geometric measure theory)
- M. Filoche, Ecole Polytechnique (condensed matter physics)
- R. Friend, Cambridge (organic semiconductors)
- D. Jerison, MIT (harmonic analysis, PDE)
- Y. Meyer, ENS-Cachan (harmonic analysis)
- J. Speck, UCSB (GaN semiconductors)
- C. Weisbuch, UCSB and Ecole Polytechnique (semiconductors)

and our many collaborators.

A world full of waves

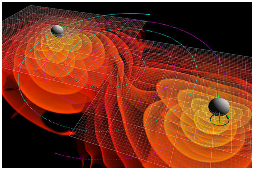
Mechanical

Fluid

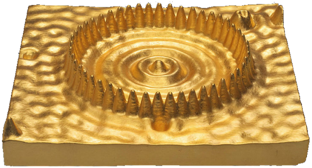


Electromagnetic

Acoustic



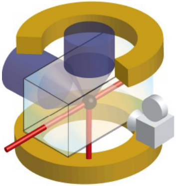
Gravitational

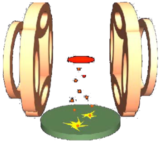


Matter

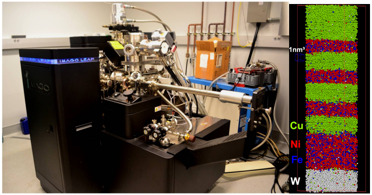
Add technology which makes the invisible visible…

- Manipulate individual atoms with laser cooling: a few billionths of a degree above absolute zero
  [Aspect lab, Institut d'Optique]

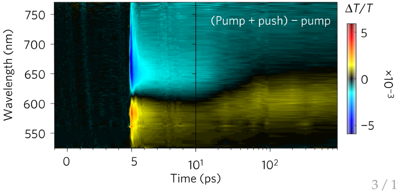




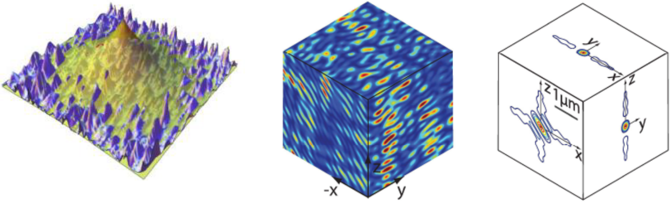
- Map out materials atom-by-atom: samples of 10,000-10,000,000 nm^3 with billions of atoms
  [Speck & Weisbuch lab, UCSB]



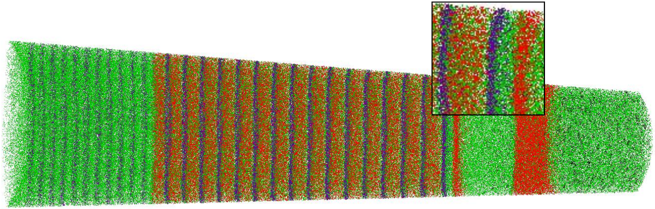
- Pump-push-probe electroabsorption: 100 fs $= 10^{-13}$ s migration of holes from higher to lower energy sites
  [Friend lab, Cambridge]



And we find disorder everywhere

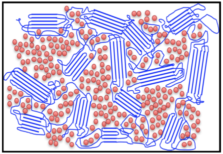


Anderson localization in Bose-Einstein condensate [Aspect lab]



Atomic map of InGaN semiconductor [Speck lab]

Mixed donor-acceptor morphology in an organic solar cell [Friend lab]



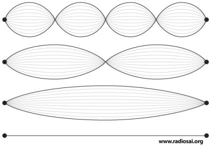
String vibration

Any string vibration is a linear combination of "harmonics" – eigenfunctions which solve

$$\begin{cases} -\frac{d^2}{dx^2}\varphi = \lambda\varphi, \\ \varphi(0) = \varphi(1) = 0 \end{cases}$$

λ are the corresponding eigenvalues (AKA energies)

Here we assume that the string is fixed at the ends and has length 1: $u(0) = u(1) = 0$

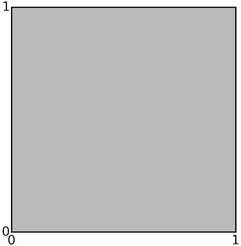


$$\varphi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

Given $\varphi(0) = \varphi(1) = 0$, we have

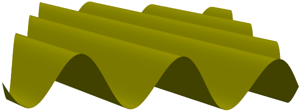
$$\lambda_n = (n\pi)^2, \quad \varphi_n = \sin(n\pi x)$$

Smooth versus disordered potential in Schrödinger equation



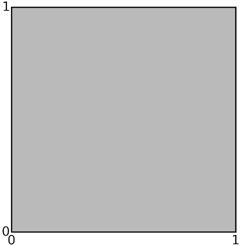
no potential

fundamental mode

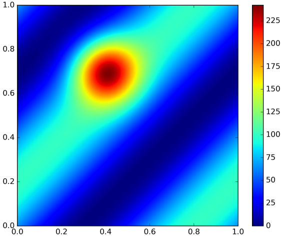


57th mode

Smooth versus disordered potential in Schrödinger equation

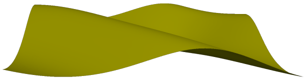


no potential

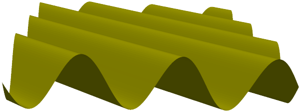


smooth potential

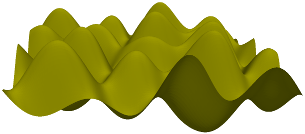
fundamental mode



fundamental mode



57th mode



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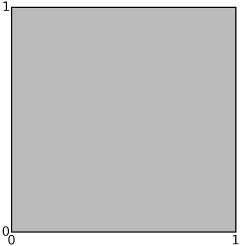
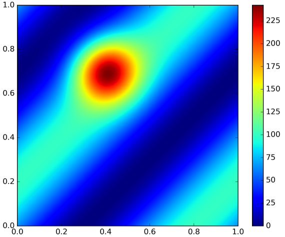
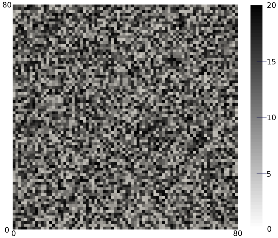


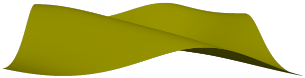
no potential



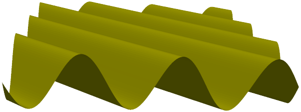
smooth potential



fundamental mode



fundamental mode



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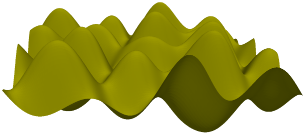

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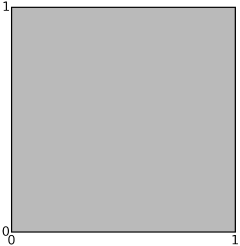
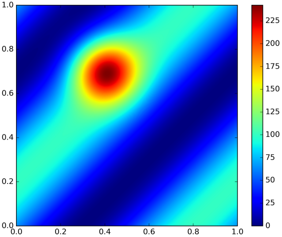
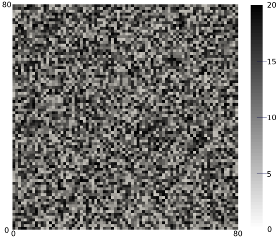


no potential



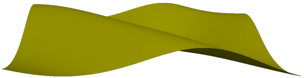
smooth potential



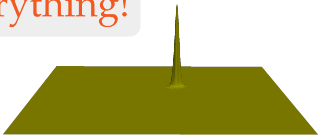
random potential

Disorder changes everything!

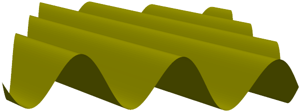
fundamental mode



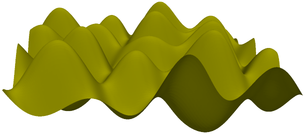
fundamental mode



fundamental mode



57th mode



57th mode

57th mode

The main goal: basic questions

We seek a quantitive, deterministic understanding of the mechanisms of wave localization so we can answer such questions as:

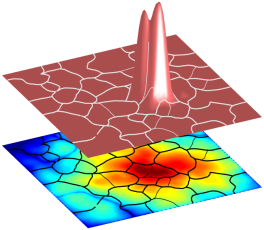
- When and where do eigenfunctions localize?
- How many localize?
- What are the size and shapes of their supports?
- What are the associated eigenvalues?

More generally, for Schrödinger and far more complex systems, we want to:

- Determine wave behavior in a given disordered environment.
- Infer the disordered environment by observing wave behavior.
- Design the environment to obtain desired or optimal wave behavior.

Take on the perspective of a wave

The main hero:
THE LANDSCAPE



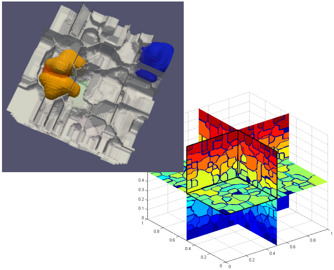
A hidden landscape that waves recognize and obey

- born of the equation but invisible to the naked eye
- contains both spatial and spectral information

The goal is to

Discover and master this landscape in order to

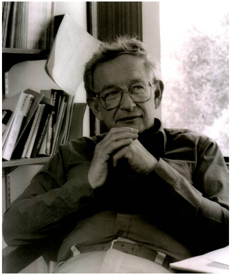
- understand
- predict
- manipulate
- govern
- and, ultimately, design matter waves



Curves/surfaces of the landscape vs. eigenfunctions

Anderson localization

The localization of Schrödinger eigenfunctions with random potential was discovered by Philip Anderson in his Nobel-prize-winning work of 1958.



> *Unfortunately, electron localization was devilishly hard to confirm... experimental observations are sparse and covered with disputes and controversies.*
>
> – Lagendijk, van Tiggelen, Wiersma, *50 Years of Anderson Localization*, 2009

> *Most theoretical work [7-9] predicts [the critical exponent]* $\mu = 1$*, but there is also a prediction of* $\mu = 1/2$ *[10]. Numerical simulation [11] gives* $\mu = 2/3\ldots$
>
> – I. Shlimak, *Is Hopping a Science?*, 2015

Anderson localization

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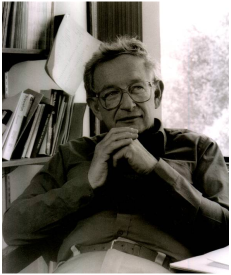


Very few believed [localization] at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it.

– Philip W. Anderson, *Nobel Lecture*, 1977

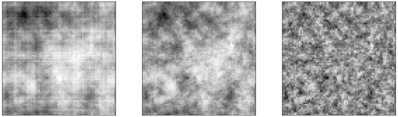
Localization beyond Anderson

Localization occurs in many, many situations of interest.

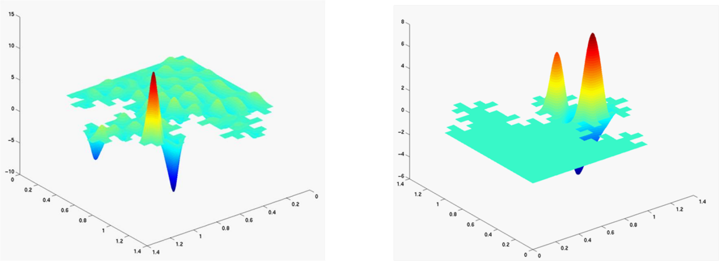
- localization in 2D, diffusion in 3D, higher dimensions
- correlations
- singular distributions (e.g., Bernoulli)
- deterministic disorder (almost Mathieu, quantum Hall effect)
- localization by geometry (e.g., Bernoulli with values 0 and $+\infty$)

Bernoulli potential

Correlated Gaussian field potential



Disorder through geometry



High-order localized Dirichlet modes of a 2D domain with fractal-like boundary

Fractal® Wall Acoustic Barrier, Filoche et al.

Localization in mathematics

A collection of rather different phenomena:

- semiclassical analysis
- mathematical physics
- probability
- PDEs
- geometric measure theory
- harmonic analysis

... to mention only a few

The landscape function

Theorem (Filoche & Mayboroda 2012)

For H be any elliptic operator (possibly with rough coefficients, complex boundary, ...), define the landscape function u by the equation

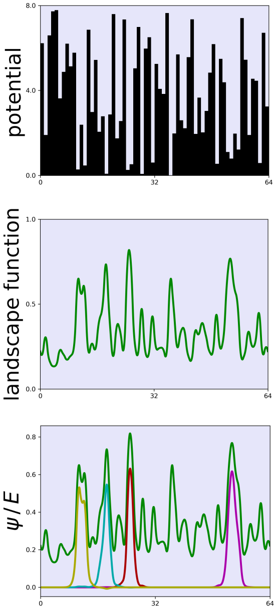
$$Hu = 1$$

plus boundary conditions. Then, any normalized eigenpair (E, ψ),

$$H\psi = E\psi, \quad \|\psi\|_{L^\infty} = 1,$$

satisfies the pointwise bound

$$|\psi(x)| \leq E\, u(x).$$



A different perspective: the effective potential

Arnold, David, Filoche, Jerison, Mayboroda, 2015–2018:

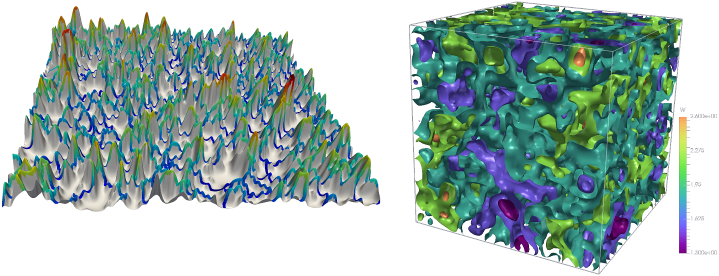
linear equation $\Longrightarrow$ nonlinear control

$W = 1/u$ is an *effective potential* which is often confining. The new eq.

$$-\frac{1}{u^2}\nabla \cdot (u^2 \nabla \phi) + \frac{1}{u}\phi = E\phi$$

has exactly the same eigenvalues as the Schrödinger equation.

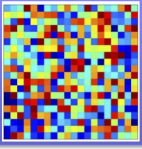
$Hu = 1 \quad \Longrightarrow \quad$ enhanced *Agmon-type distance* $\rho_{1/u} \quad \Longrightarrow \quad$ exp decay



2D and 3D effective potential $\frac{1}{u}$ for Bernoulli V

Localization by disorder: $L = -\Delta + V$

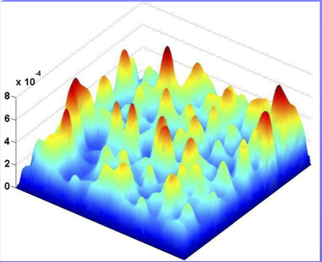
Operator ***L***



The localization scheme

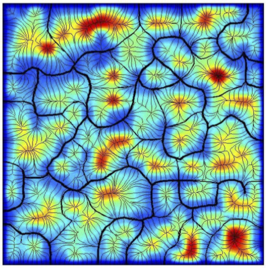
Quantum states in a random potential

$$L\psi(x, y) = E\psi(x, y)$$



Main new idea: **the landscape** ***u***

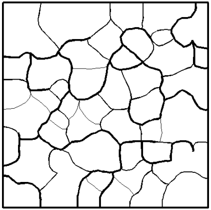
$$L\,u(x, y) = 1$$



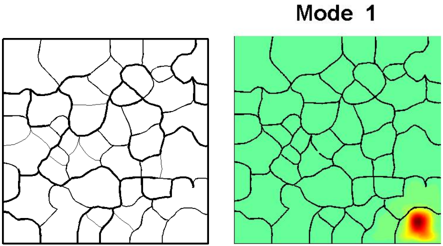
Valley network of ***u***
(black curves)

Filoche & Mayboroda, PNAS, 2012

Anderson Localization: valleys and quantum states



Anderson Localization: valleys and quantum states



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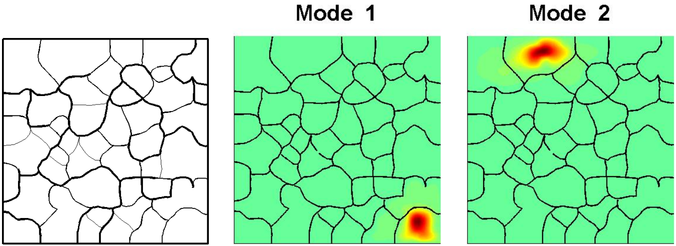

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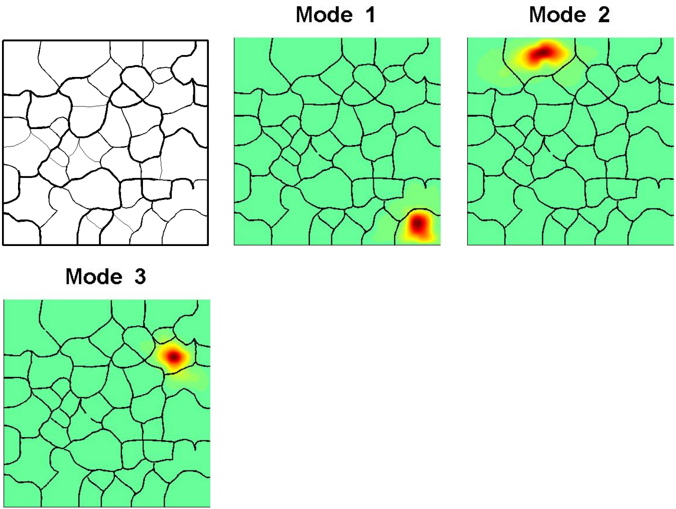

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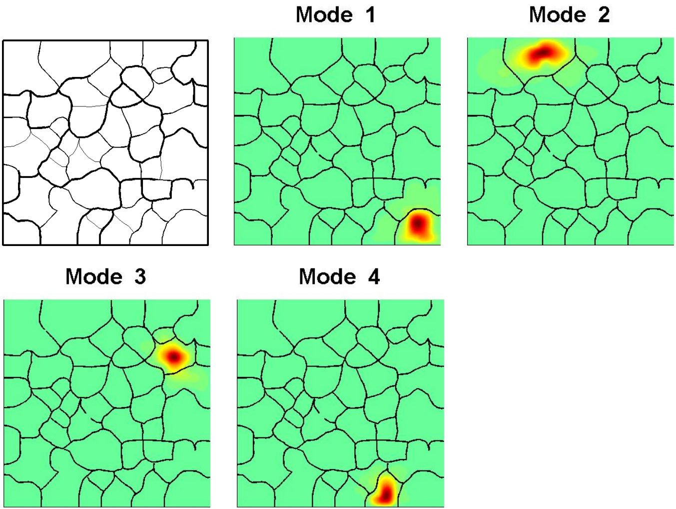
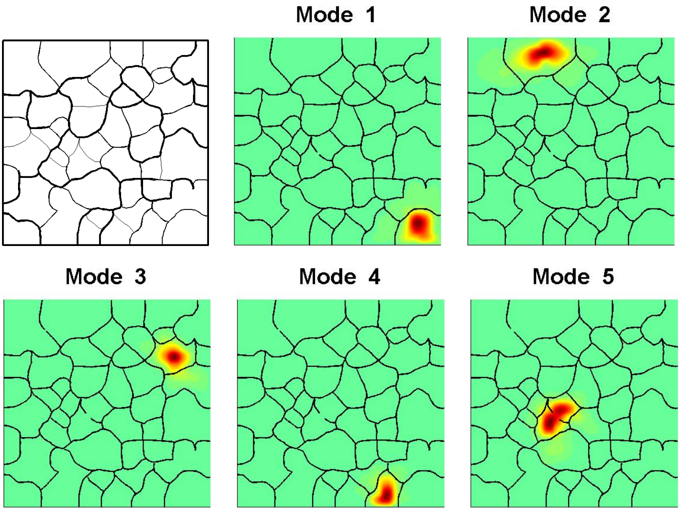
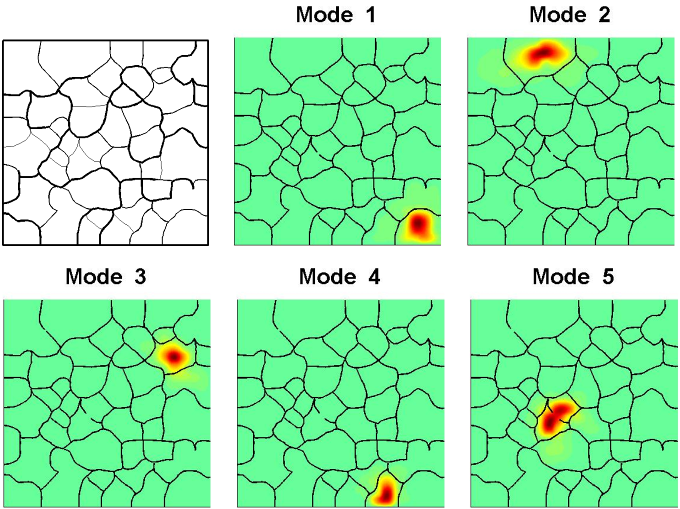


Anderson Localization: valleys and quantum states



Anderson Localization: valleys and quantum states



Movie

$1/u$ is an effective potential
(joint with Arnold, David, Filoche, Jerison)

- The watershed basins associated to its minima predict the localization regions.

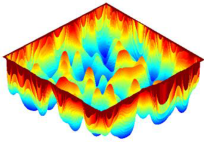
$$\frac{\psi(x)}{\max \psi} \leq E u(x)$$

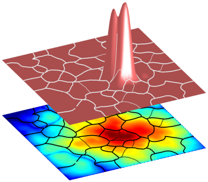
- Its crests induce exponential decay.

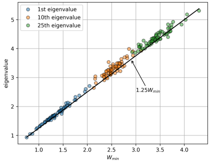
$$\psi(x) \sim \exp\left\{ -\rho_{\left(\frac{1}{u}-E\right)_+}(x) \right\}$$

- Its well depths predict the energy levels.

$$\min \frac{1}{u} \approx \left(1+\frac{n}{4}\right) E$$


effective potential $1/u$


watershed lines of $1/u$ and an eigenfunction

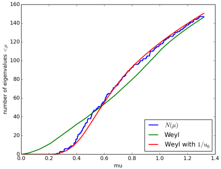


min $1/u$ vs E

Reading the localization landscape

- Modifying Weyl's law by replacing the true potential with the effective one predicts the density of states with astonishing accuracy.

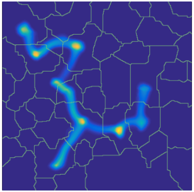
$$N(E) \approx Vol\left\{\xi^2 + \frac{1}{u} \leq E\right\}$$



Blue: reality; green: old Weyl; red: new Weyl

- It gives access to transport properties (hopping).

$$\langle \psi_1 | e^{-i\vec{q}\cdot\vec{r}} | \psi_2 \rangle \approx \int e^{-i\vec{q}\cdot\vec{r}} e^{\rho_{1,\frac{1}{u}}(\vec{r}) + \rho_{2,\frac{1}{u}}(\vec{r})}\, d\vec{r}$$



Transport