MATHEMATICAL FRONTIERS

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Mathematical Sciences & Analytics

MATHEMATICAL FRONTIERS
2019 Monthly Webinar Series, 2-3pm ET

February 12: *Machine Learning for Materials Science**

March 12: *Mathematics of Privacy**

April 9: *Mathematics of Gravitational Waves*

May 14: *Algebraic Geometry*

June 11: *Mathematics of Transportation*

July 9: *Cryptography & Cybersecurity*

August 13: *Machine Learning in Medicine*

** Recordings posted*

September 10: *Logic and Foundations*

October 8: *Mathematics of Quantum Physics*

November 12: *Quantum Encryption*

December 10: *Machine Learning for Text*

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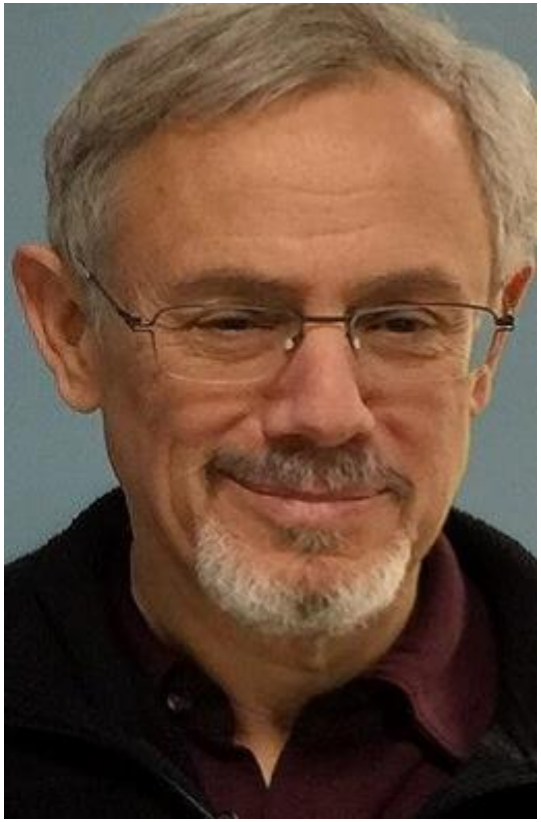
MATHEMATICAL FRONTIERS
Mathematics of Gravitational Waves

Manuela Campanelli,
Rochester Institute of Technology

Thomas Baumgarte,
Bowdoin College



Mark Green,
UCLA (moderator)

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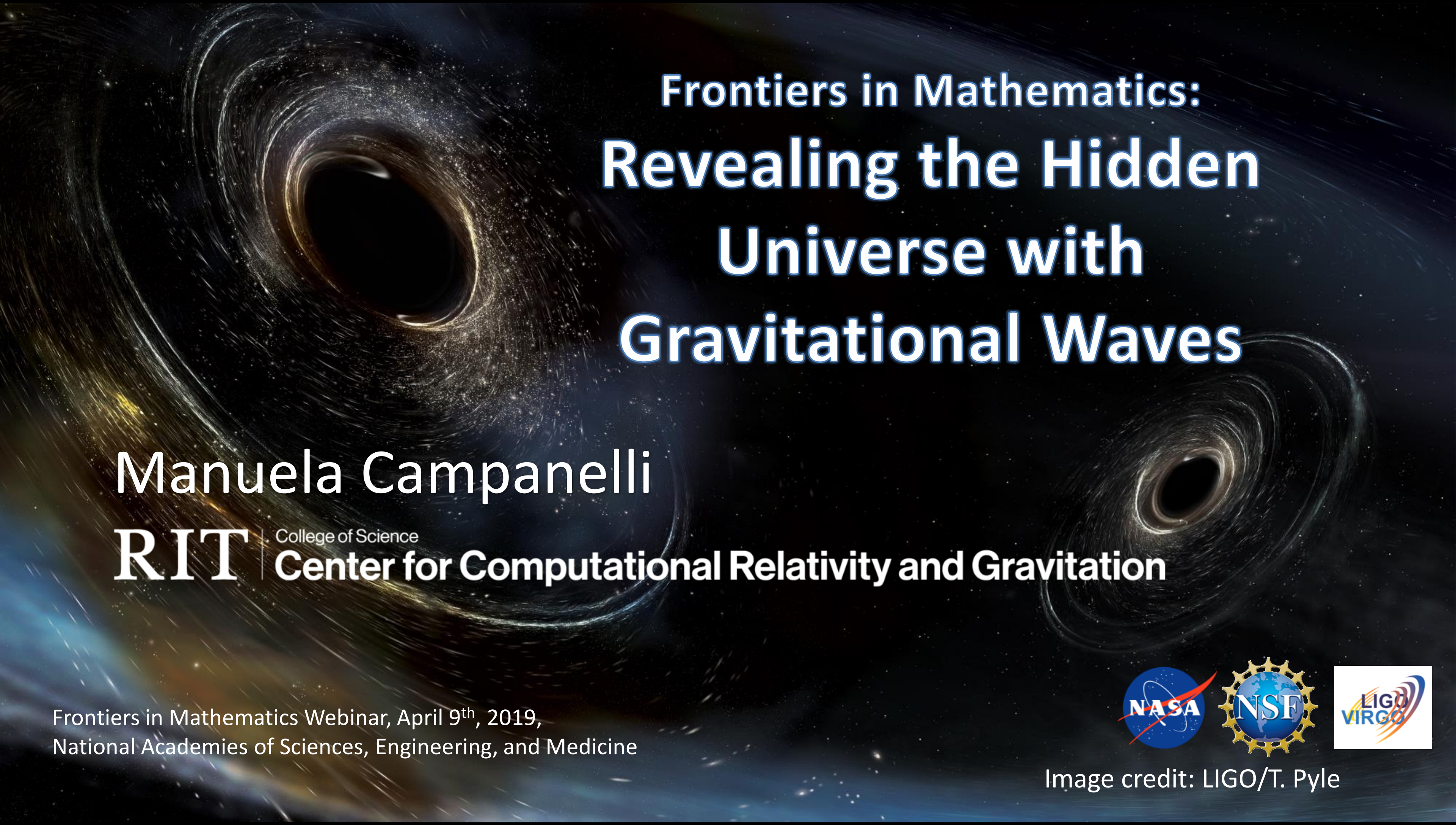
MATHEMATICAL FRONTIERS
Mathematics of Gravitational Waves

Manuela Campanelli,
Rochester Institute of Technology

Professor of Mathematics
Director, Center for Computational Relativity and Gravitation

Revealing the Hidden Universe with Gravitational Waves



Frontiers in Mathematics:

Revealing the Hidden Universe with Gravitational Waves

Manuela Campanelli

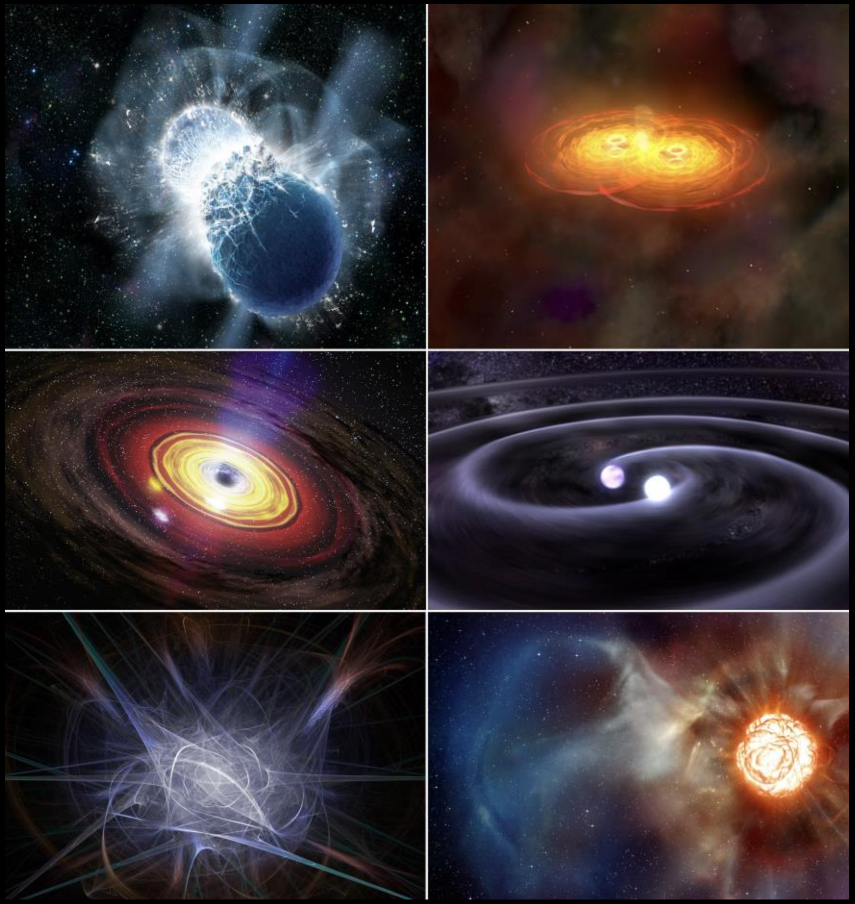
RIT | College of Science
Center for Computational Relativity and Gravitation

Frontiers in Mathematics Webinar, April 9th, 2019,
National Academies of Sciences, Engineering, and Medicine

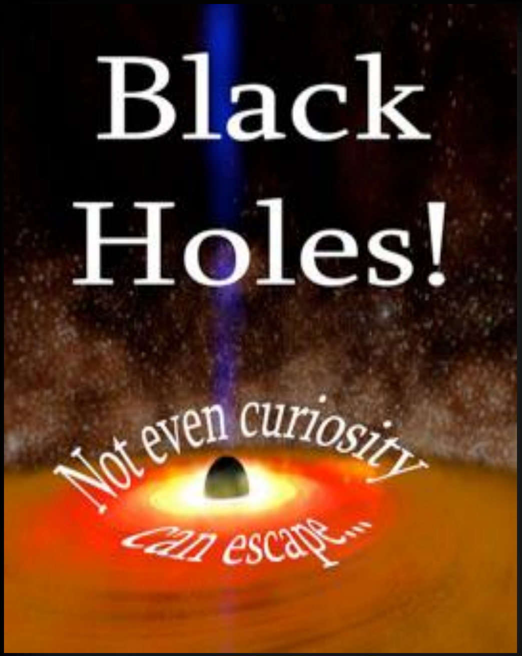
Image credit: LIGO/T. Pyle

What are we interested in?

- Understanding the most extreme astrophysical objects!
- Dangerous distant places where gravity is very strong, matter is very dense and magnetic fields are very extreme!



The "Invisible" Universe …



Regions of space where gravity is so intense that it prevents all matter and **even light** from escaping!

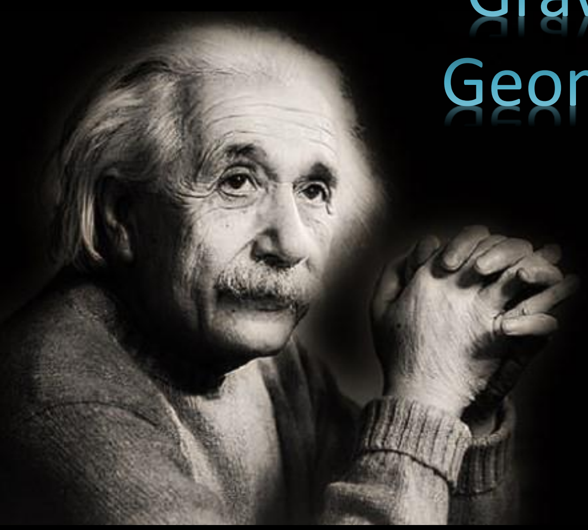
The Theory of General Relativity

1915

Space-Time Curvature = Matter-Energy

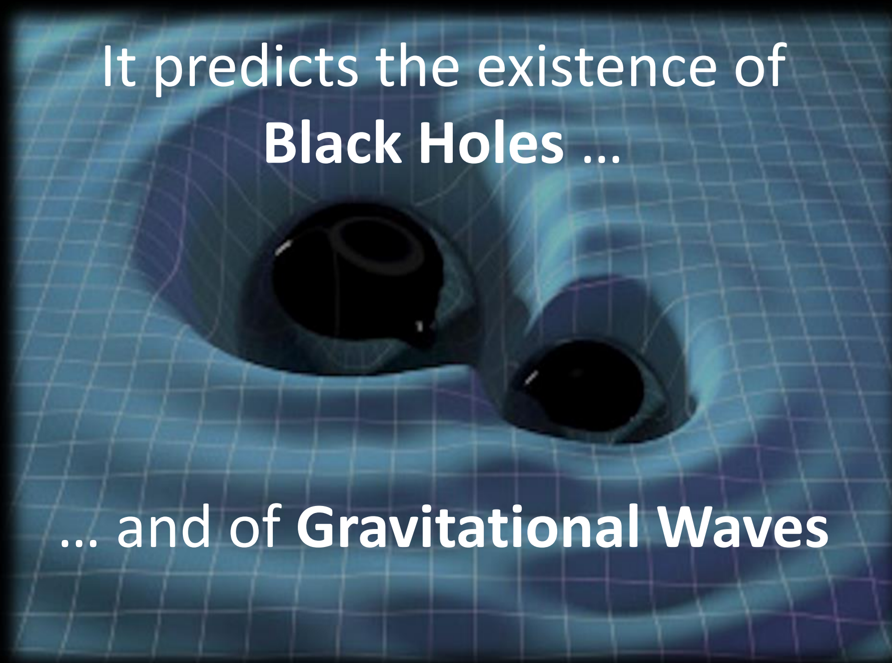
$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Gravity is Geometry!



“Space tells matter how to move and matter tells space how to curve”

English translation by John Archibald Wheeler

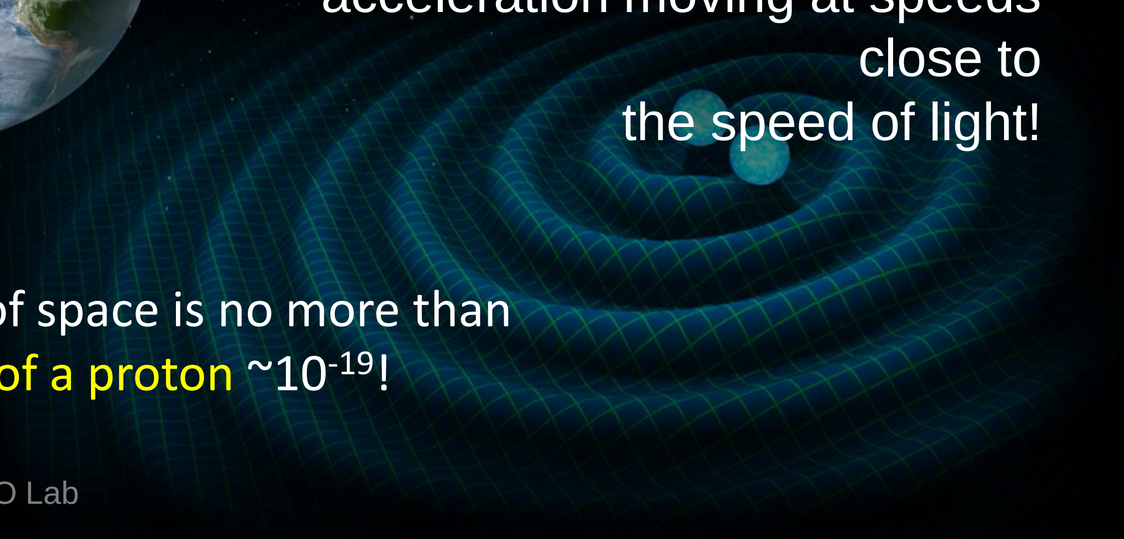


Things that can ripple the space-time!

Gravitational waves stretch and compress space-time itself!

Colliding black holes and/or neutron stars and explosions of stars!

A lot of mass in very rapid acceleration moving at speeds close to the speed of light!

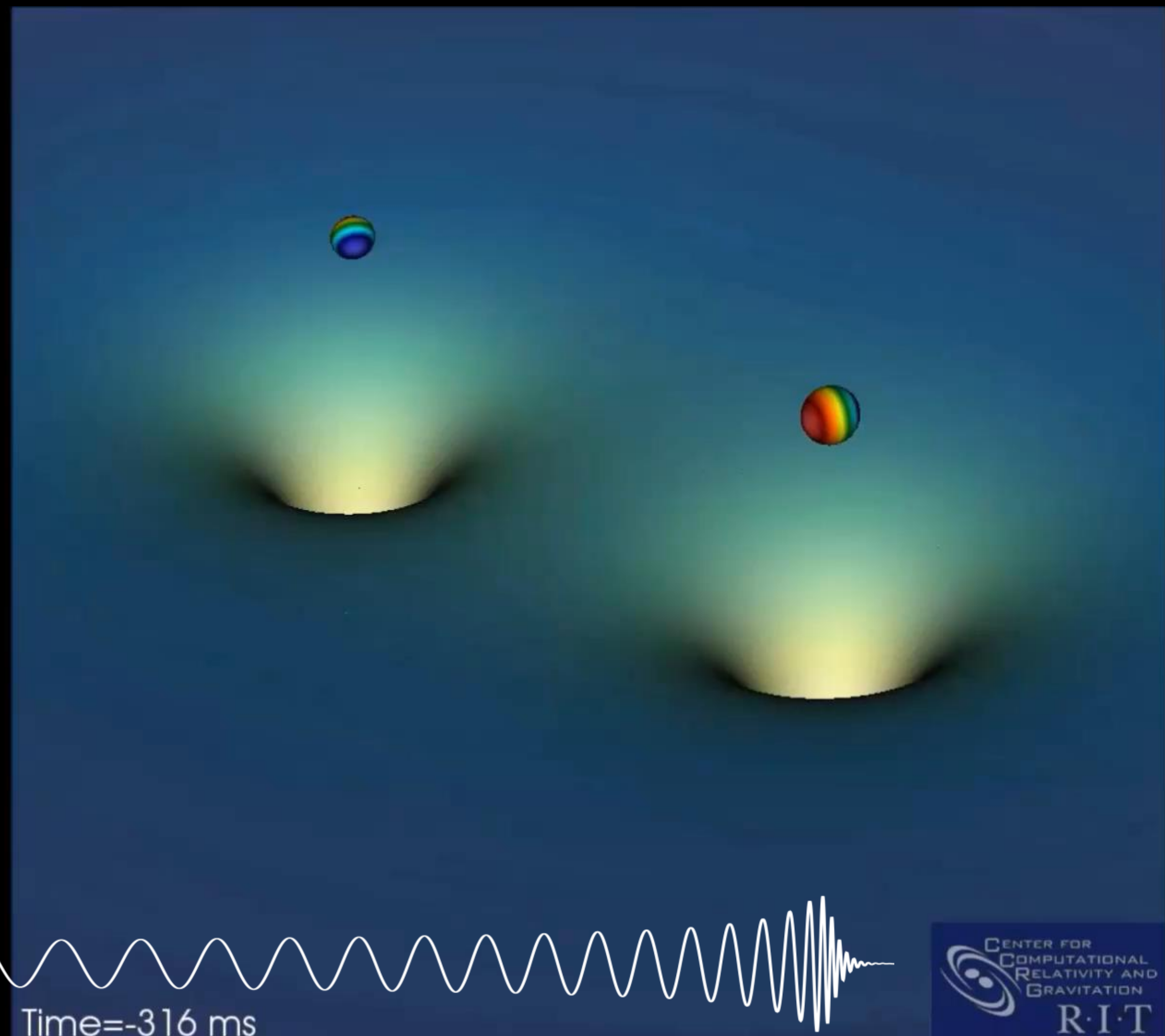


This stretching and squeezing of space is no more than one ten thousandth the width of a proton ~10^{-19}!

Animation created by R. Hurt, Caltech/MIT/LIGO Lab

Numerical Simulation of a Binary Black Hole Collision

Solving the Einstein's equations with the use of supercomputers!



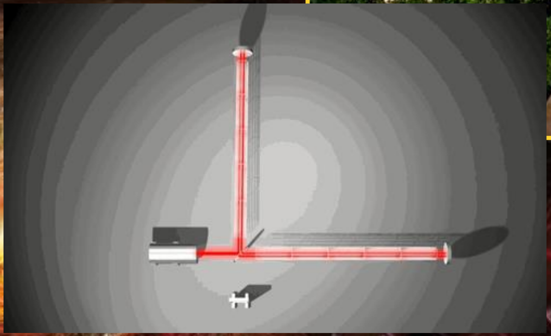
Video Credits: Nicole Rosato RIT

Traveling at 60% of the speed of light!

Ripping space-time apart as they go!

Catching the waves from these Cosmic Collisions

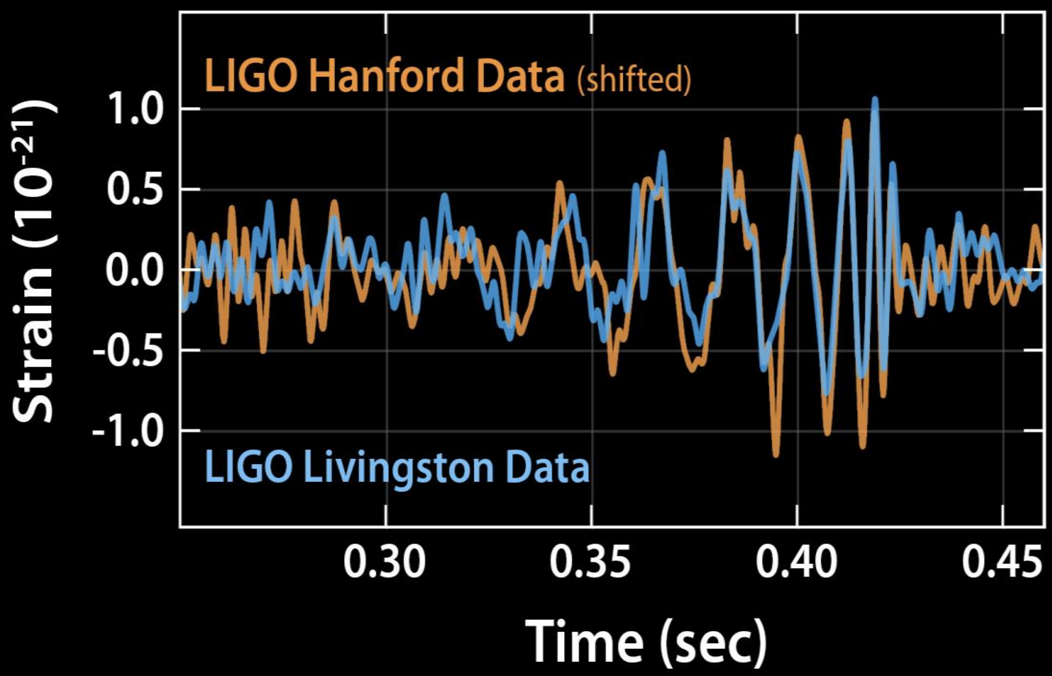
These waves are entirely different spectrum than light …



You need a special ruler to measure the stretching and squeezing of space …

The First Black Hole Binary Merger

Was detected on September 14, 2015



GW150914 occurred at a distance of more than one billion light years away!

Original Black Holes:
36 + 29 solar masses
Final Black Hole:
62 solar masses

Abbott et al, Phys. Rev. Lett. **116**, 061102 (2016)

How Powerful This Was?

3 solar masses missing …

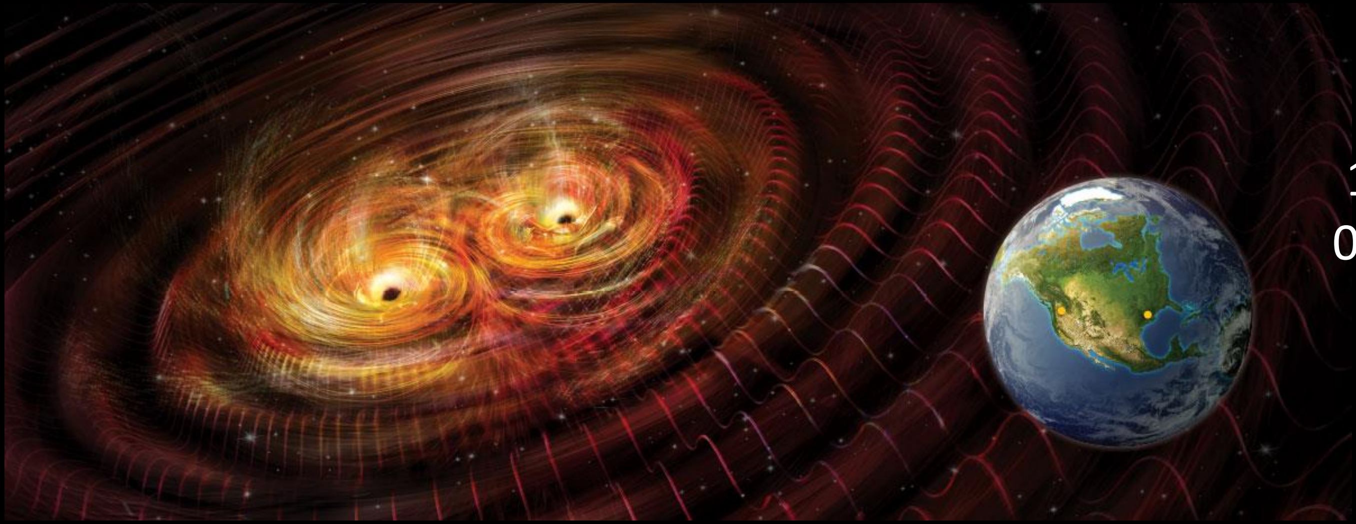
1 Solar Mass = 333,000 Earth Masses!

$M c^2 = E$

3 solar masses x c^2 = 4.5 x 10^{47} Joules

in 0.3 sec = 1 x 10^{48} Watts

radiated away in gravitational waves



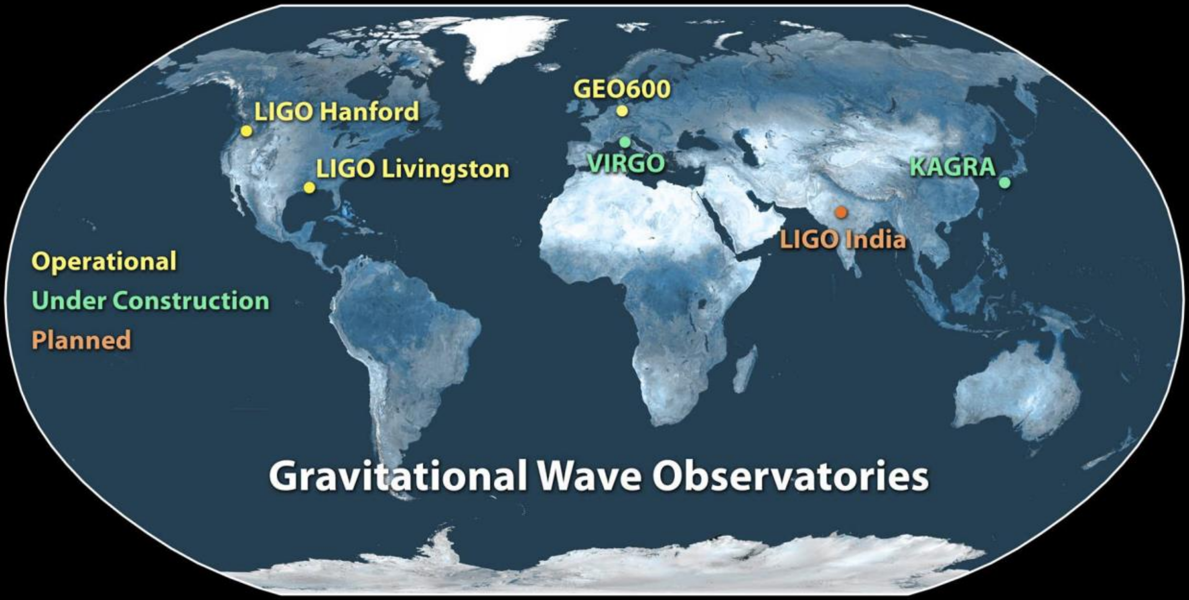
19 000 000 000 000 000
000 000 000 000 000 000
000 000 000 000,
60W light bulbs

This is more than 10 times the combined the light power of every star and galaxy in the observable Universe!

2017 Nobel Prize in Physics

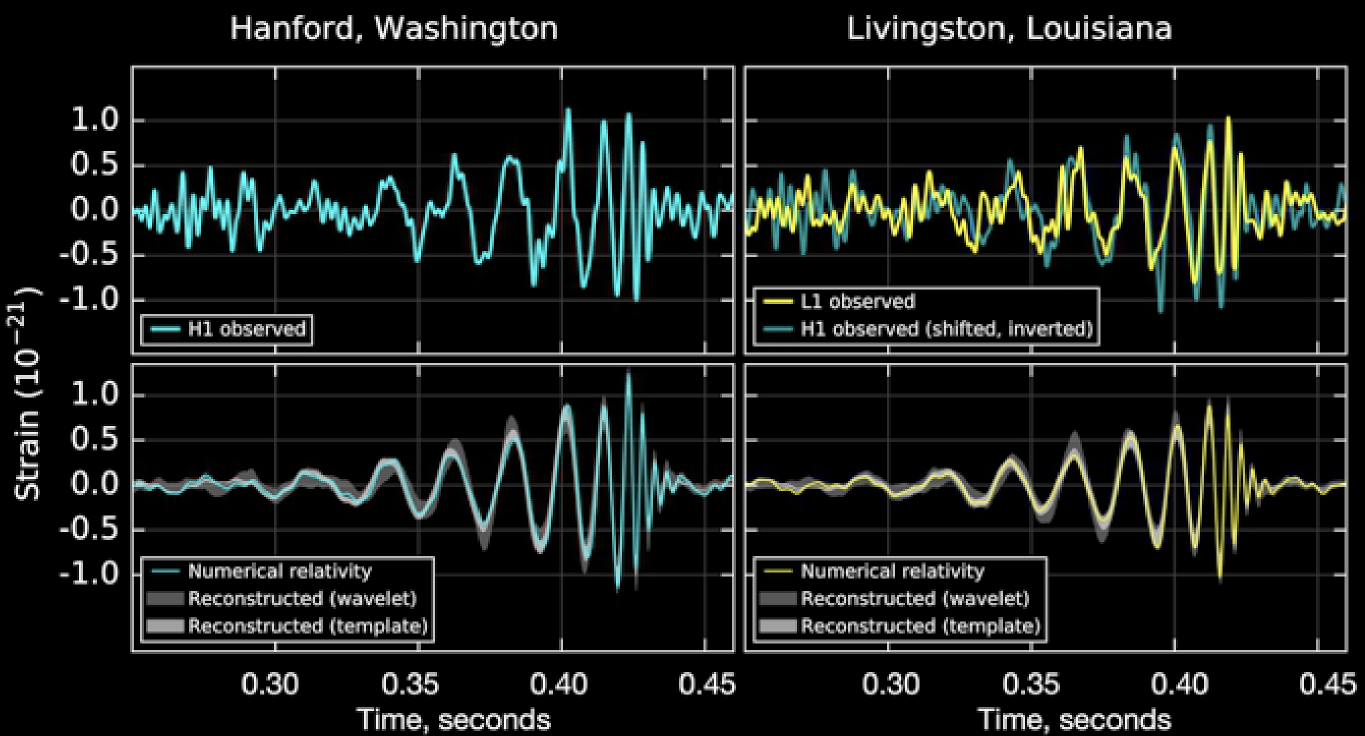
"for decisive contributions to the LIGO detector and the observation of gravitational waves"

Big scientific discovery requires big collaborations



LIGO/Virgo scientific collaboration (LVC):
1000 scientists, 16 countries

How well does the predicted signal match the data ?



Abbott et al, Phys. Rev. Lett. **116**, 061102 (2016)

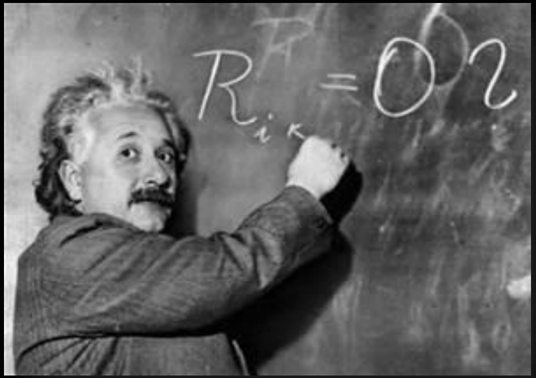
Solving the Einstein's Equations

A system of nonlinear, coupled, partial differential equations with hundreds of terms!

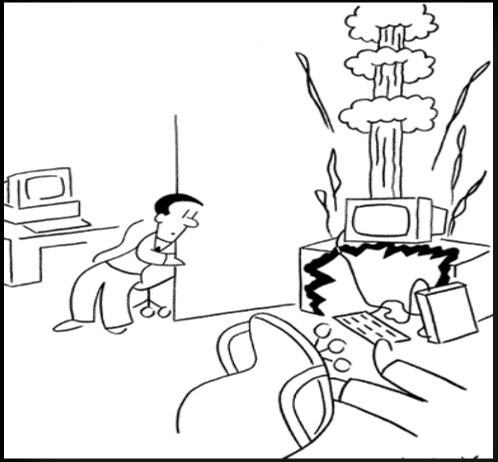
$$
\begin{aligned}
\partial_0 \tilde{\gamma}_{ij} &= -2\alpha \tilde{A}_{ij}, \\
\partial_t \chi &= \frac{2}{3}\chi\left(\alpha K - \partial_a \beta^a\right) + \beta^i \partial_i \chi, \\
\partial_0 \tilde{A}_{ij} &= \chi\left(-D_i D_j \alpha + \alpha R_{ij}\right)^{TF} + \\
& \quad \alpha\left(K \tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right), \\
\partial_0 K &= -D^i D_i \alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right), \\
\partial_t \tilde{\Gamma}^i &= \tilde{\gamma}^{jk}\partial_j \partial_k \beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j \partial_k \beta^k + \beta^j \partial_j \tilde{\Gamma}^i - \\
& \quad \tilde{\Gamma}^j \partial_j \beta^i + \frac{2}{3}\tilde{\Gamma}^i \partial_j \beta^j - 2\tilde{A}^{ij}\partial_j \alpha + \\
& \quad 2\alpha\left(\tilde{\Gamma}^i{}_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}\partial_j \phi - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K\right),
\end{aligned}
$$

The **BSSN formalism** developed by Baumgarte and collaborators in 1999

- They also change character depending on the freedom we have to explicitly write them down!
- And black holes have singularities!



Not a simple feat!



A lot of computer horse power!

This requires very advanced numerical algorithms, which translates into in hundred thousands lines of code!

And the processing power of several petabytes (one thousand million million bytes — 1,000,000,000,000,000 bytes) of information at once!

Brief History of Simulations!

Kip S. Thorne, 2017 Nobel Prize lecture.

It took more than four decades for researchers to solve the problem

But in 2005, we finally did it!

Catalogs of pre-calculated waveforms!

To compare them to LIGO data, and extract information about the black holes!

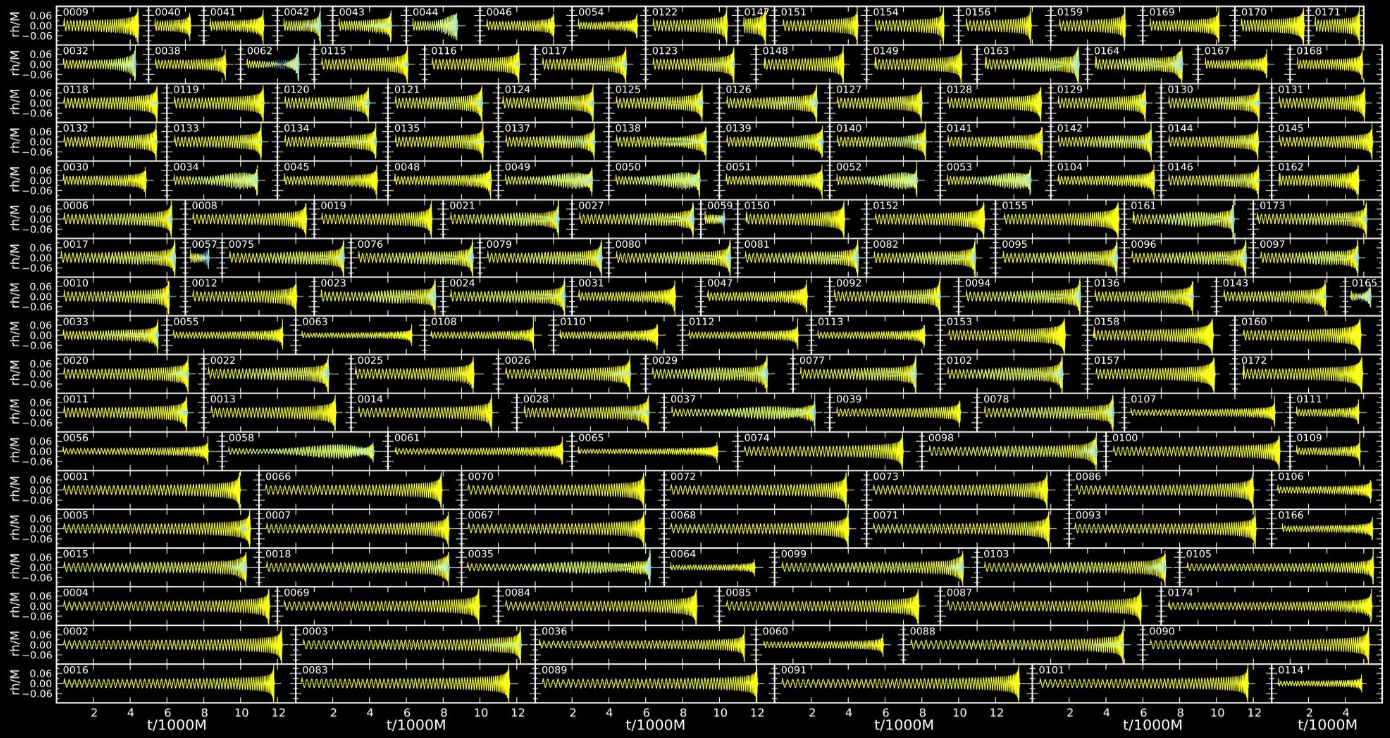
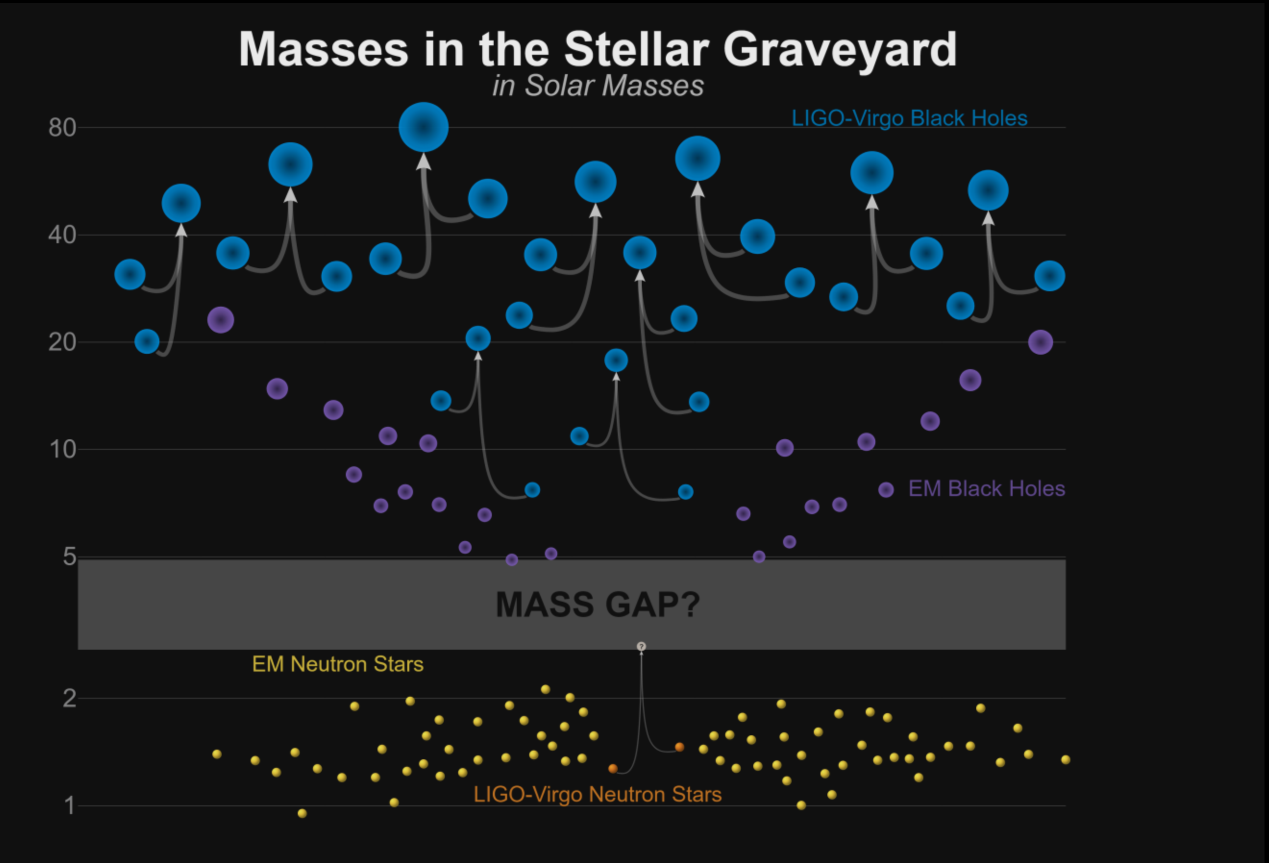


Figure courtesy the SXS Collaboration

How many have LIGO/Virgo detected so far?

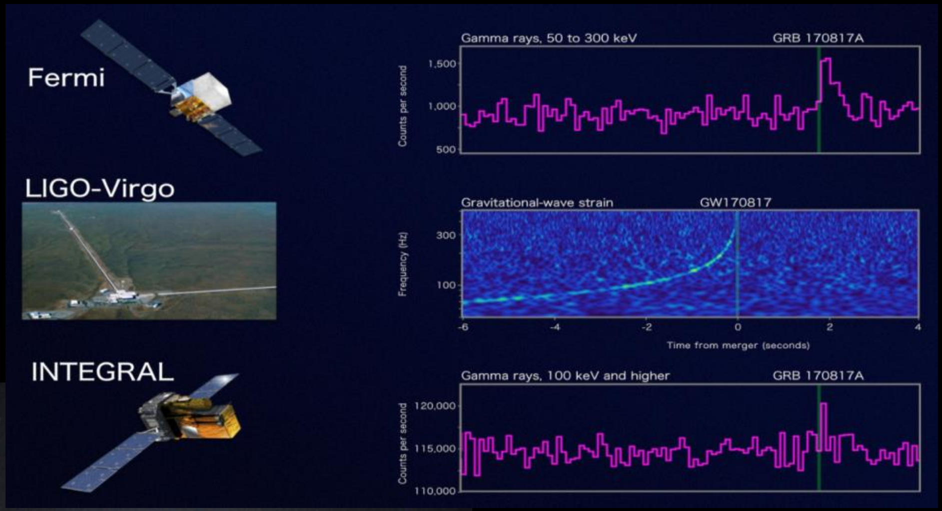


1 Solar Mass = 333,000 Earth Masses!

The LIGO/Virgo detectors found 10 black hole binary mergers and one neutron star merger!

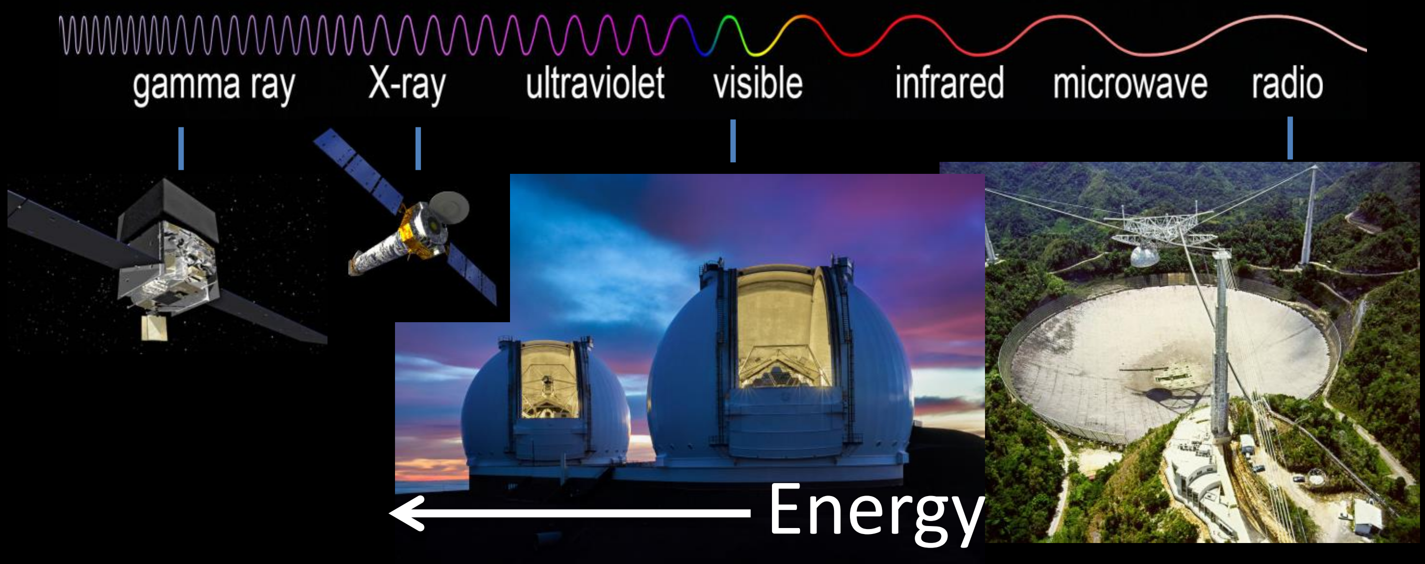
But stay tuned as LIGO/Virgo just restarted taking new data!

About that Neutron Star Merger …



More than 70 astronomy follow-up observations!

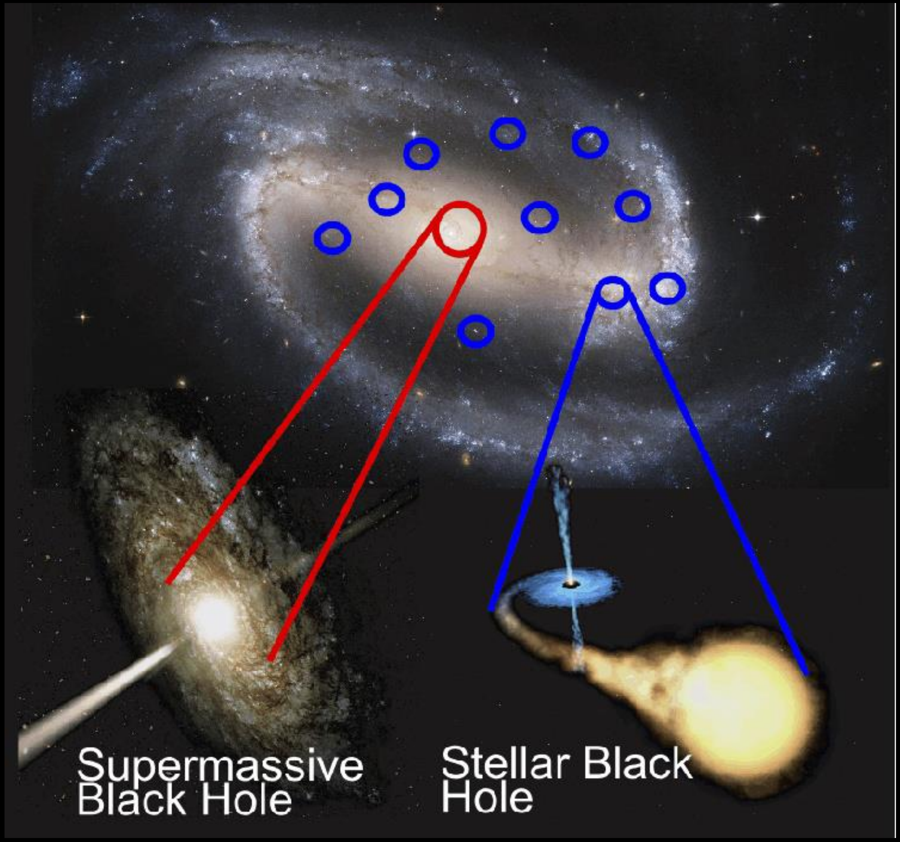
The Dawn of a New Kind of Astronomy



$$E = M c^2$$

Black Holes are quite universal objects

Their masses can range from few times the one of our Sun to billions of Suns!



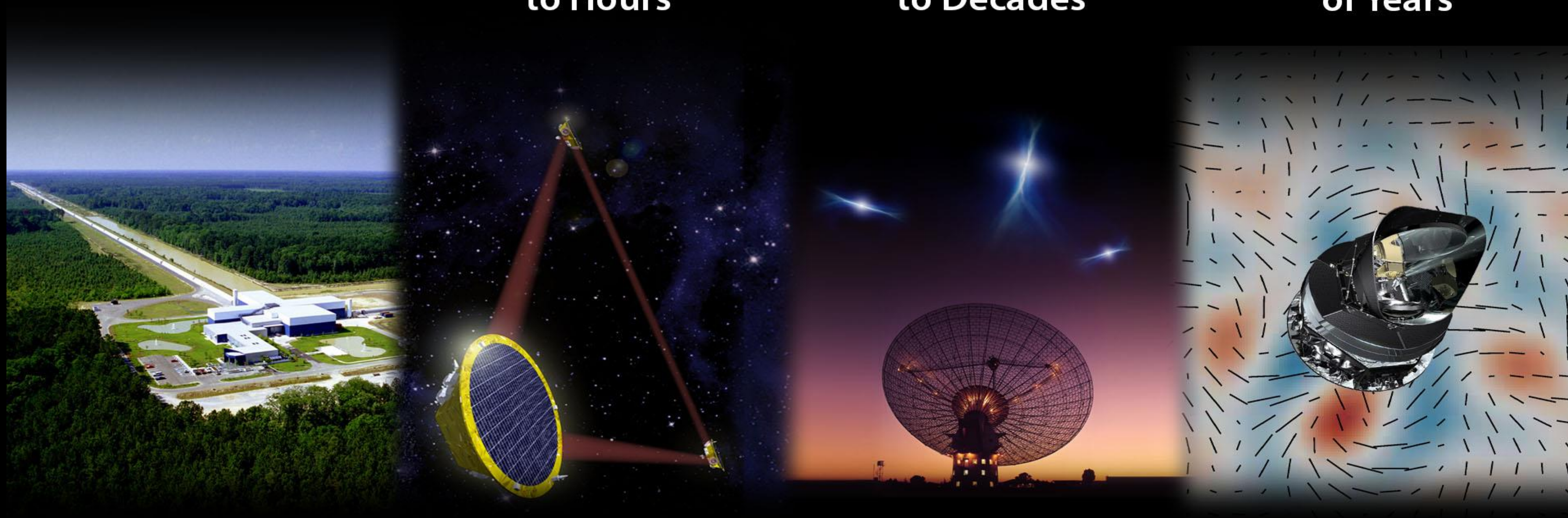
~2,400 billion Solar Masses

1 billion Solar Masses = 1,000,000,000 Solar Masses
1 Solar Mass = 333,000 Earth Masses

Searching for All Black Hole Mergers

Gravitational Wave Periods

Milliseconds	Minutes to Hours	Years to Decades	Billions of Years

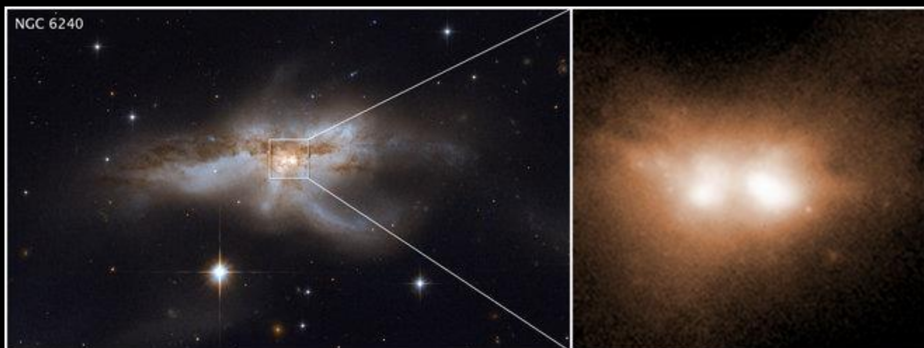


Solving the mystery of supermassive black holes at the center of galaxies

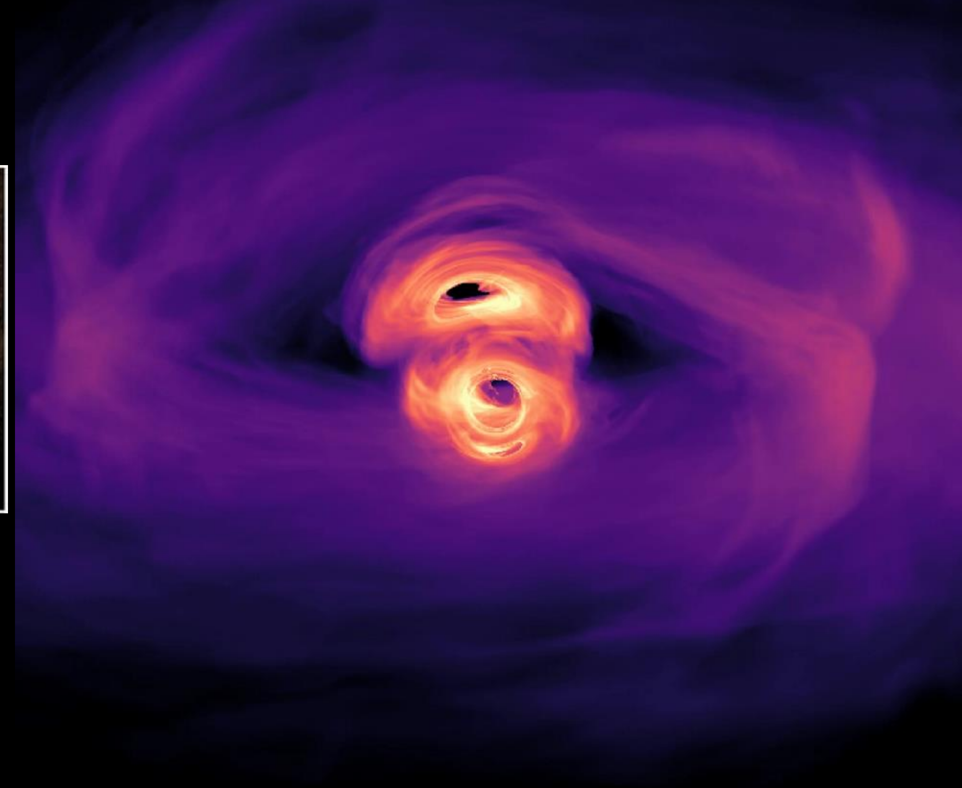
Because galaxies do merge, the supermassive black holes at their cores should merge too …

Supermassive black holes at the center of galaxies are surrounded by accreting hot gas and emit powerful radio jets!

What happens when supermassive black holes collide?



We are working on the modeling and observations of these monster collisions!



Credits: RIT/NASA simulation

More Sophisticated, Lengthy, Mathematics!!

Space-Time Curvature = Matter-Energy

$$G_{\mu\nu} = 8\pi\, T_{\mu\nu}$$

General Relativity. + Magneto-Hydrodynamics

$$\mathrm{G}_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi \tfrac{G}{c^4} T_{ab}$$

$$\mathrm{R}_{ab} = \textstyle\sum_{c=1}^{4} R^{c}_{acb};\ R = g^{ab} R_{ab}$$

$$\mathrm{R}^{c}_{acb} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{cd} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc} \leftarrow 1 \text{ Derivative}$$

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ad} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right) \leftarrow 1 \text{ more Derivative}$$

$$\frac{\partial}{\partial t}\mathbf{q}(\mathbf{P}) \quad + \quad \frac{\partial}{\partial x^i}\mathbf{F}^i(\mathbf{P}) \quad = \quad \mathbf{S}(\mathbf{P})$$

$$\frac{\partial}{\partial t}\sqrt{-g}\begin{bmatrix} \rho u^t \\ T^t{}_t + \rho u^t \\ T^t{}_j \\ B^k \end{bmatrix} + \frac{\partial}{\partial x^i}\sqrt{-g}\begin{bmatrix} \rho u^i \\ T^i{}_t + \rho u^i \\ T^i{}_j \\ \left(b^i u^k - b^k u^i\right) \end{bmatrix} = \sqrt{-g}\begin{bmatrix} 0 \\ T^\kappa{}_\lambda \Gamma^\lambda{}_{t\kappa} - \mathcal{F}_t \\ T^\kappa{}_\lambda \Gamma^\lambda{}_{j\kappa} - \mathcal{F}_j \\ 0 \end{bmatrix}$$

$$T_{\mu\nu} = \left(\rho + u + p + 2p_m\right) u_\mu u_\nu + \left(p + p_m\right) g_{\mu\nu} - b_\mu b_\nu$$

Mass Density; Internal Energy Density; Gas Pressure; Fluid's 4-velocity; Magnetic Pressure; Magnetic 4-vector; Radiative Energy & Momentum Loss

Several physics scales from microphysics to astrophysics!
Vey long simulations and many parameters!

We have just opened a new window into the universe.

There might be surprises awaiting for us!

“The most incomprehensible thing about the Universe is that it is comprehensible!

Stay Tuned for More Soon!

MATHEMATICAL FRONTIERS
Mathematics of Gravitational Waves

Thomas Baumgarte,
Bowdoin College

William R. Kenan Professor of Physics

Einstein's Gravity and the Geometry of Black Holes

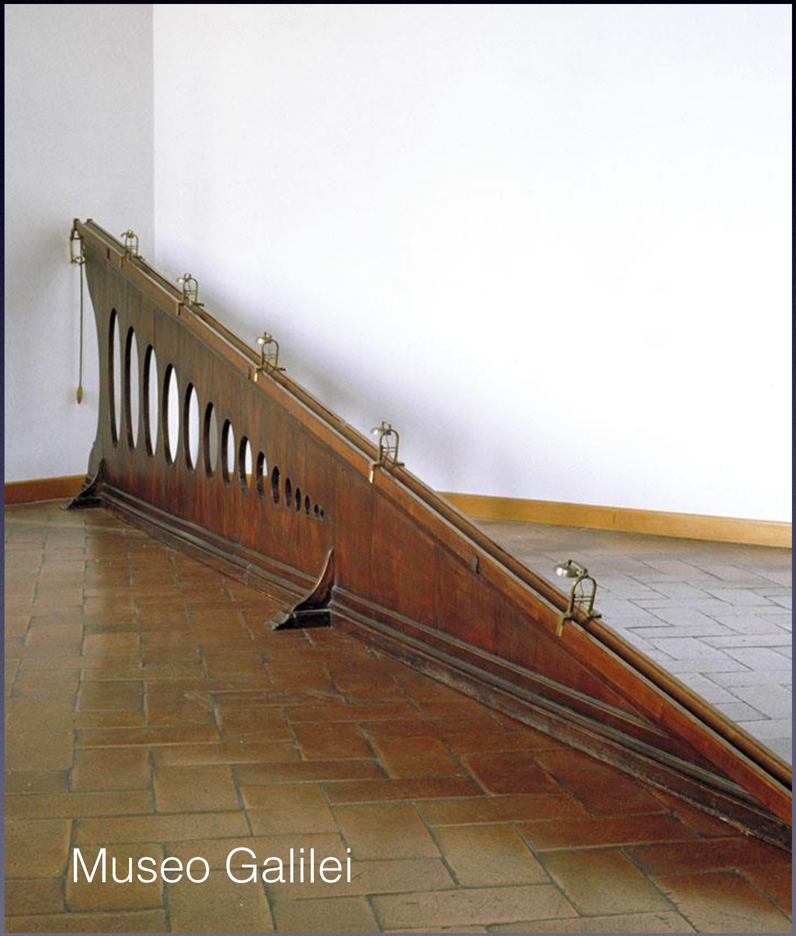
Galileo Galilei (1564-1642)

- Conducts experiments on falling objects


Justus Sustermans

Galileo Galilei (1564-1642)

- Conducts experiments on falling objects
  - All objects fall at same rate
  - Leads to *equivalence principle*

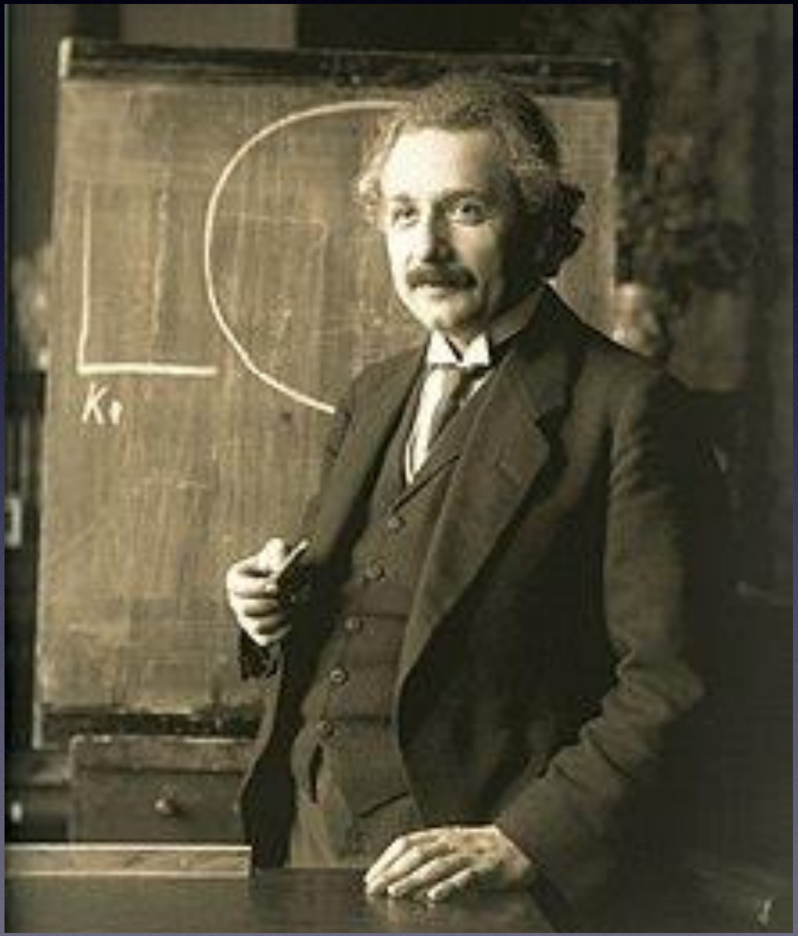


Museo Galilei

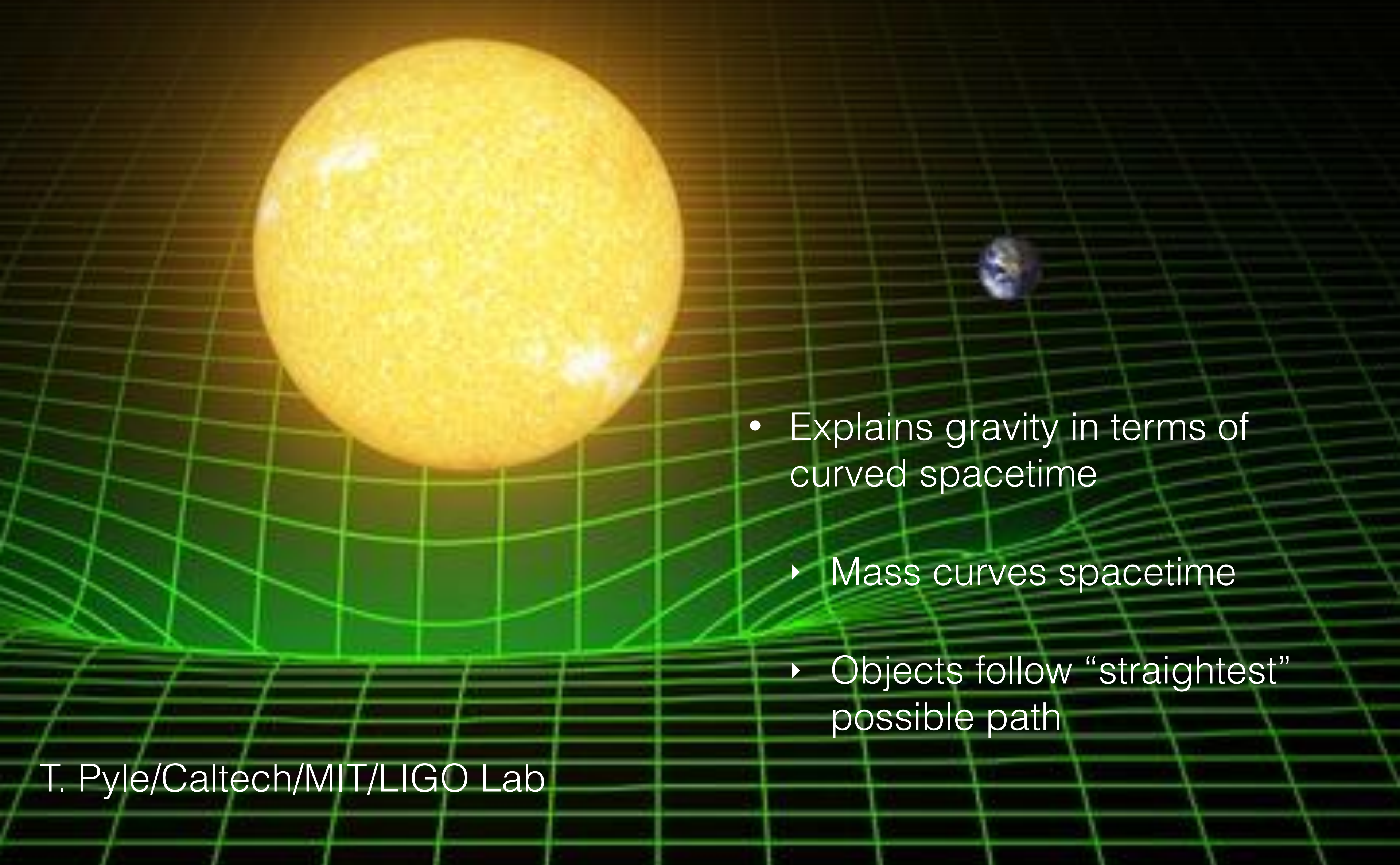
Albert Einstein (1879-1955)

- Equivalence principle leads to *General Relativity*

- Explains gravity in terms of curved spacetime



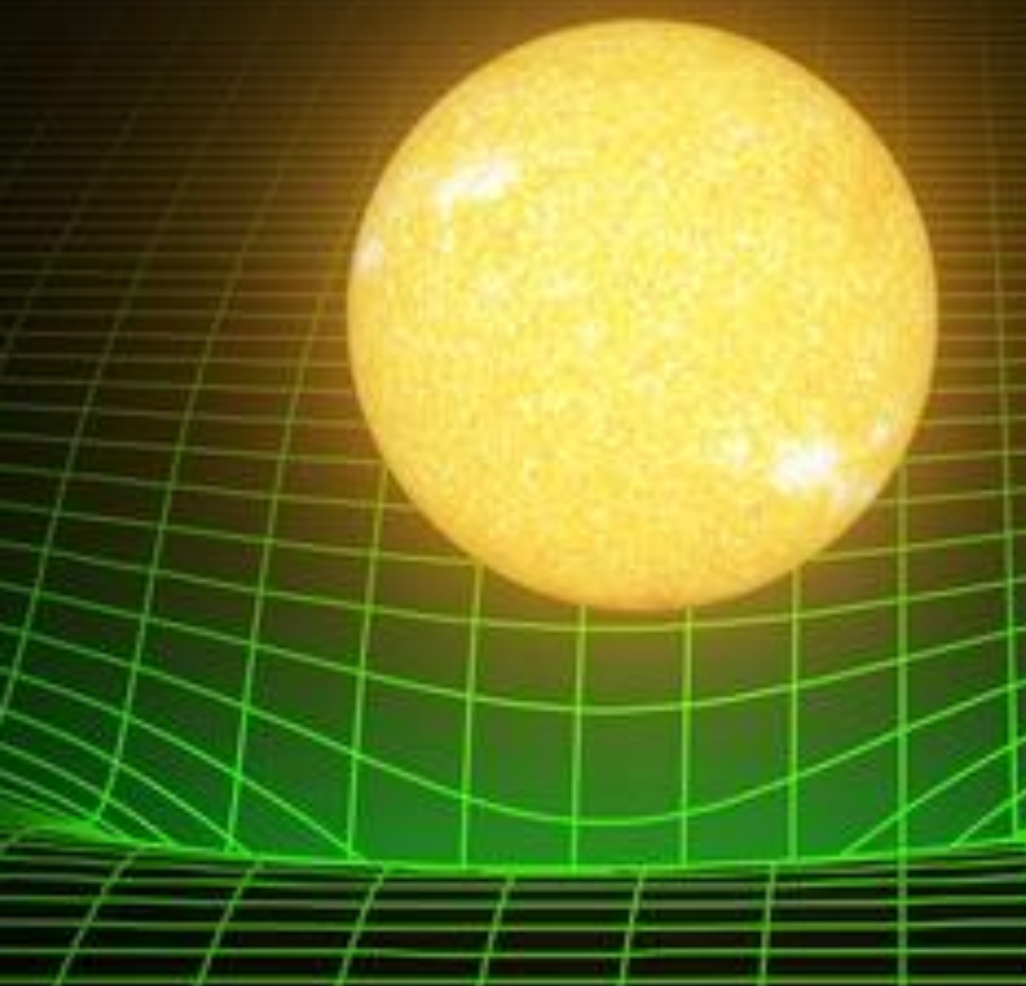
General Relativity



- Explains gravity in terms of curved spacetime
  - Mass curves spacetime
  - Objects follow “straightest” possible path

T. Pyle/Caltech/MIT/LIGO Lab

General Relativity

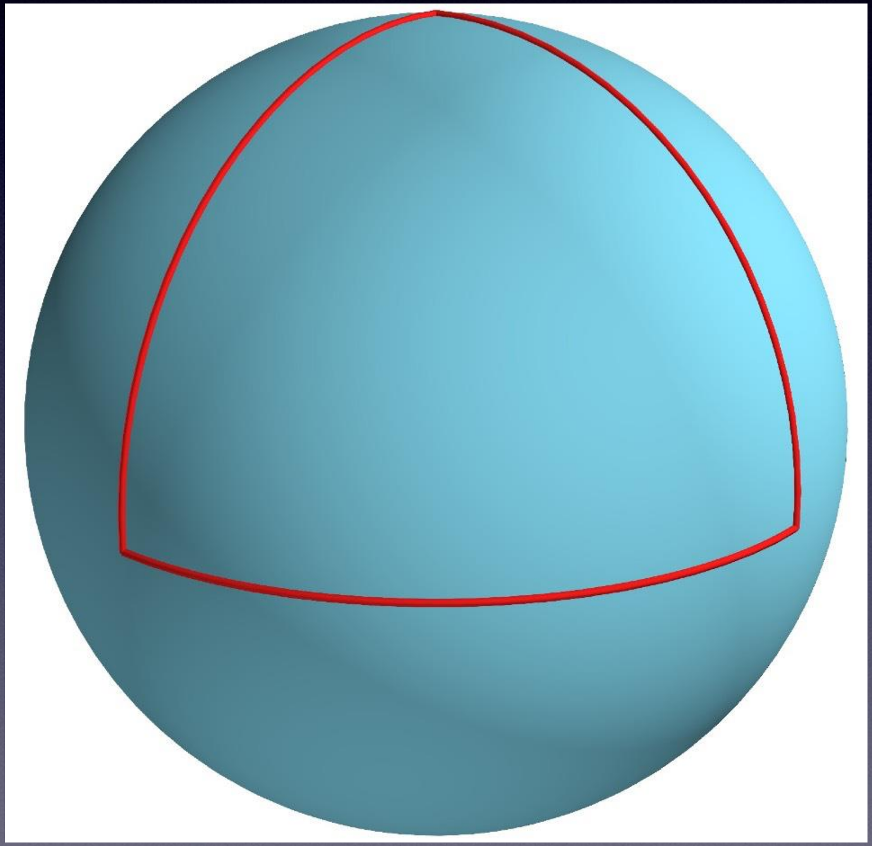


- Explains gravity in terms of curved spacetime
  - Mass curves spacetime
  - Objects follow “straightest” possible path

T. Pyle/Caltech/MIT/LIGO Lab

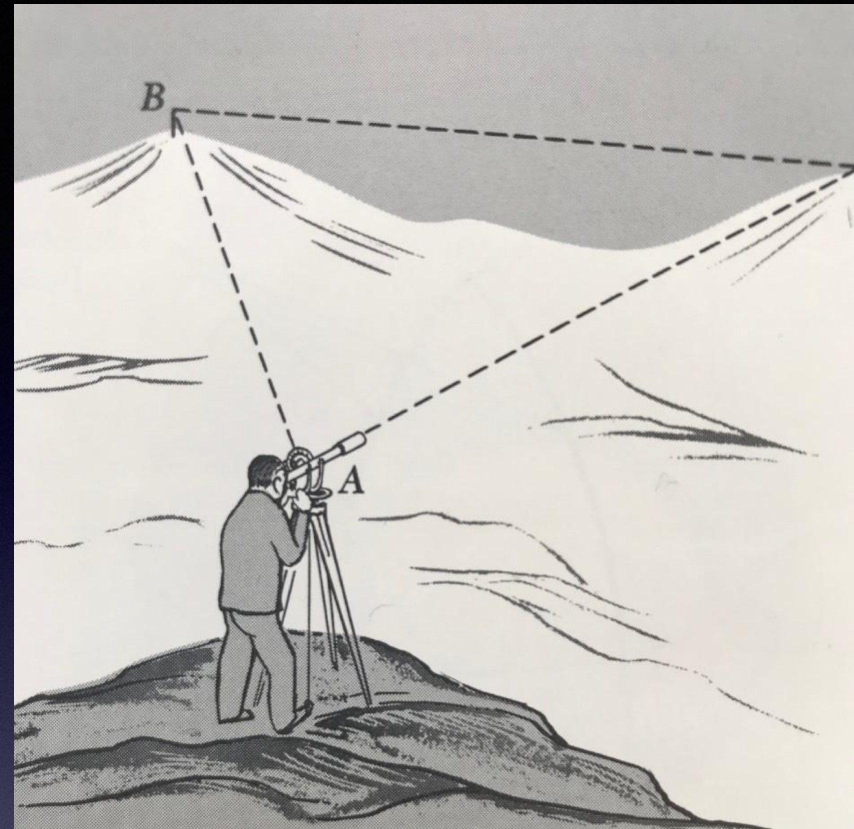
How to measure curvature?

- One option: measure interior angles of triangles



Carl
Friedrich
Gauss
(1777-1855)

Carl Friedrich Gauss (1777-1855)

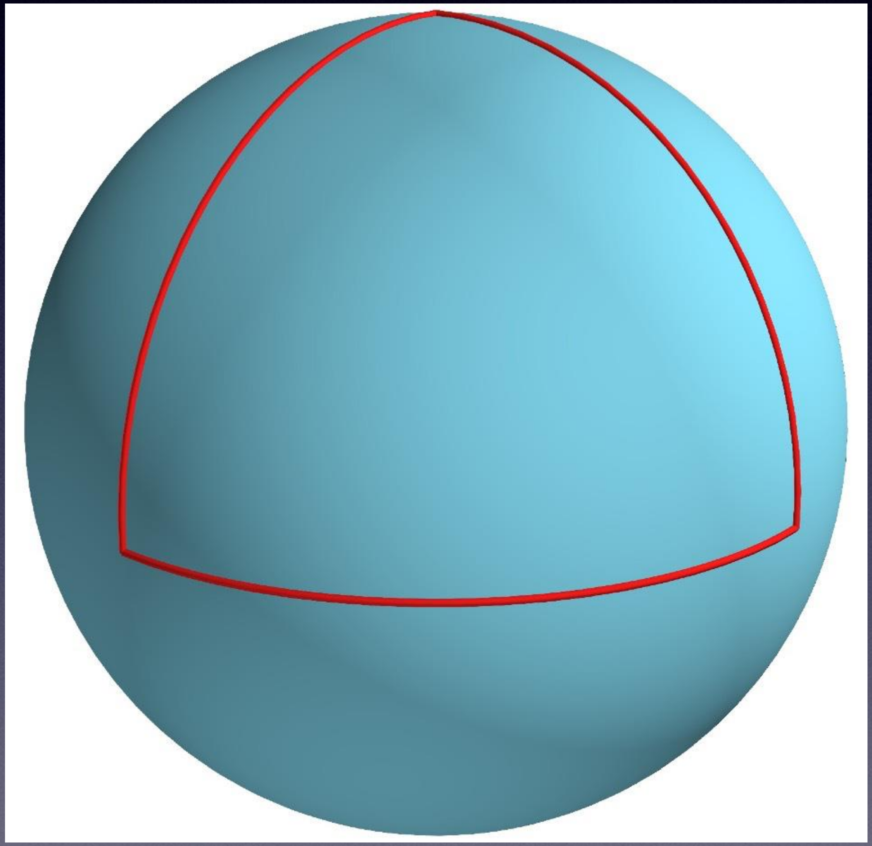


Gauss measured the angles of a triangle with vertices on three mountain tops and found no deviation from 180°, within the accuracy of his measurements.

Kittel, Knight & Ruderman, *Mechanics*

How to measure curvature?

- One option: measure interior angles of triangles
- Need *metric* to measure lengths and angles



Coordinates and distances

- Coordinates are just labels (think street numbers)
- Need *metric* to convert coordinate distances into *proper distances*
- In one dimension: scale factor
- In general: matrix (rank-2 tensor) g_{ab}: *metric*

Coordinates and distances

- Coordinates are just labels (think street numbers)
- Need *metric* to convert coordinate distances into *proper distances*
- In one dimension: scale factor
- In general: matrix (rank-2 tensor) g_{ab}: *metric*

Example: Minkowski Metric

- Metric of flat space (special relativity in Cartesian coordinates): Minkowski metric

$$g_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Then *line element*

$$ds^2 = g_{ab} dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2$$

- “Generalized Pythagoras”

How to compute Curvature (for real)?

- Compute *Riemann tensor*

$$R^{a}{}_{bcd}$$

  - Involves up to second derivatives of metric g_{ab}
  - Measures *tidal fields,* like second derivatives of Newtonian potential ϕ

Einstein’s equations

- Metric satisfies *Einstein’s field equations*

$$G_{ab} = 8\pi T_{ab}$$

where *Einstein tensor* G_{ab} computed from $R^a{}_{bcd}$

- Newtonian cousin: Poisson equation

$$\nabla^2 \phi = 4\pi\rho$$

- Left-hand sides: second derivatives of g_{ab} or ϕ
- Right-hand sides: matter sources

Karl Schwarzschild (1873-1916)

Letter to Einstein, 12/22/15:

As you see, the war has treated me kindly enough, in spite of the of heavy gunfire, to allow me to get away from it all, and take a walk in the land of your ideas.

Karl Schwarzschild (1873-1916)

Einstein's response:

I have read your paper with the utmost interest. I had not expected that one could formulate the exact solution in such a simple way. […] Next Thursday I shall present the work to the Academy…

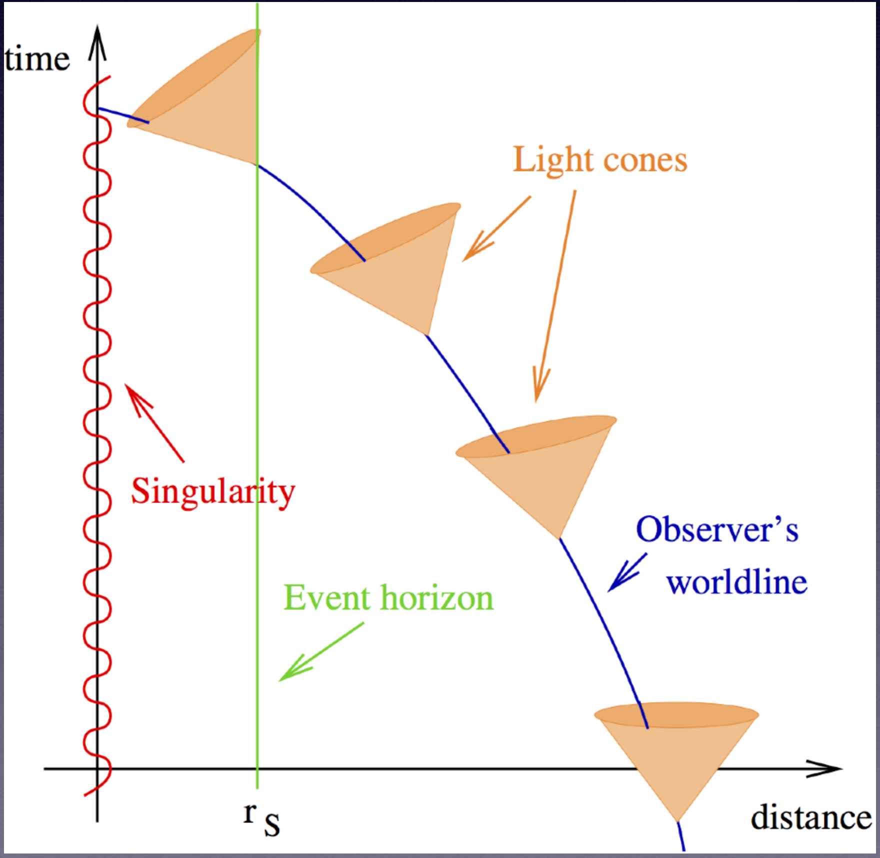
The Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Astrophysical significance not appreciated until 1960's
- Describes non-rotating black holes, characterized by
  - Curvature singularity at center:

$$R^a{}_{bcd} \longrightarrow \infty$$

  - Event horizon at $r_S = 2M$: "one-way" membrane



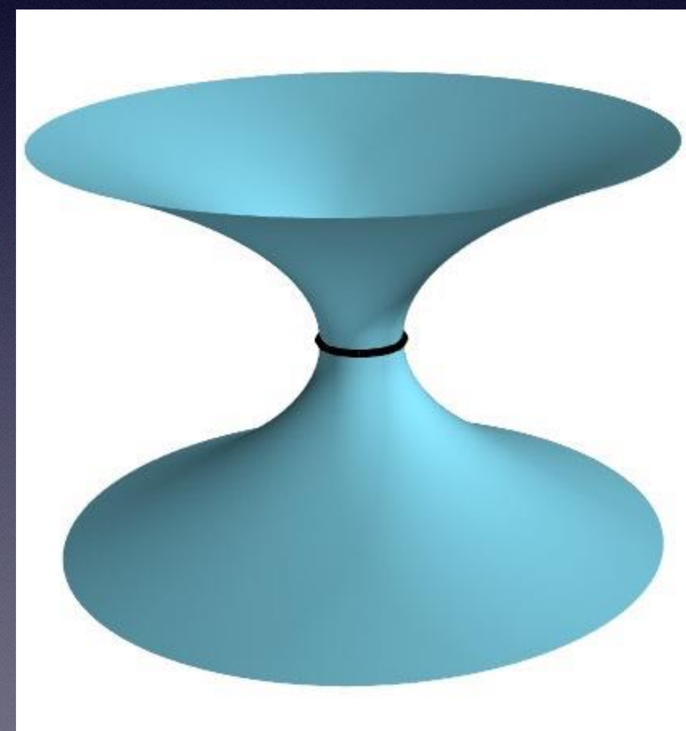
Embedding diagram

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

To visualize geometry:

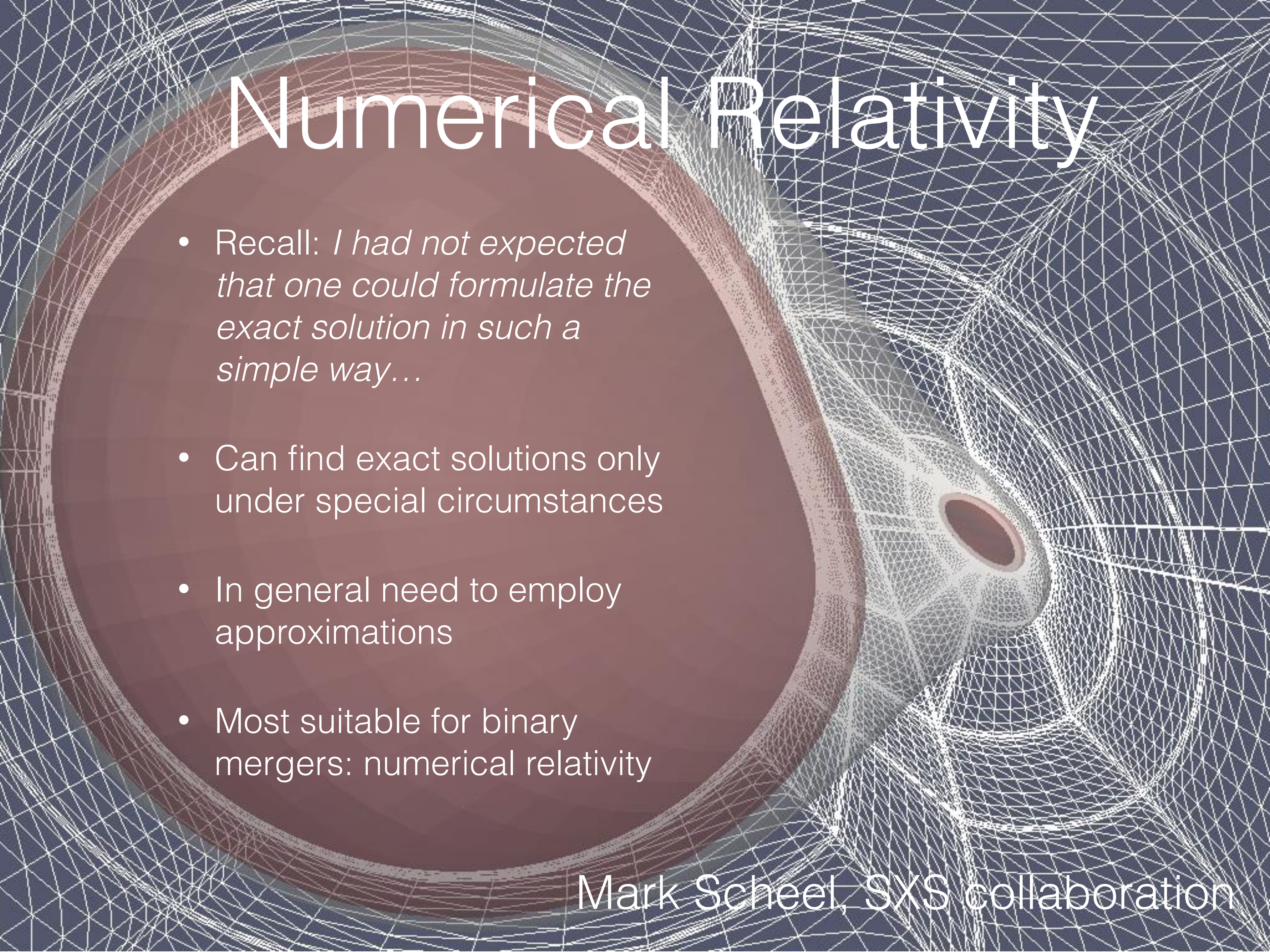
- Choose $t = const$
- Choose equatorial plane, $\theta = \pi/2$
- Embed resulting 2D surface in flat 3D space

‣ *Wormhole* geometry



Numerical Relativity

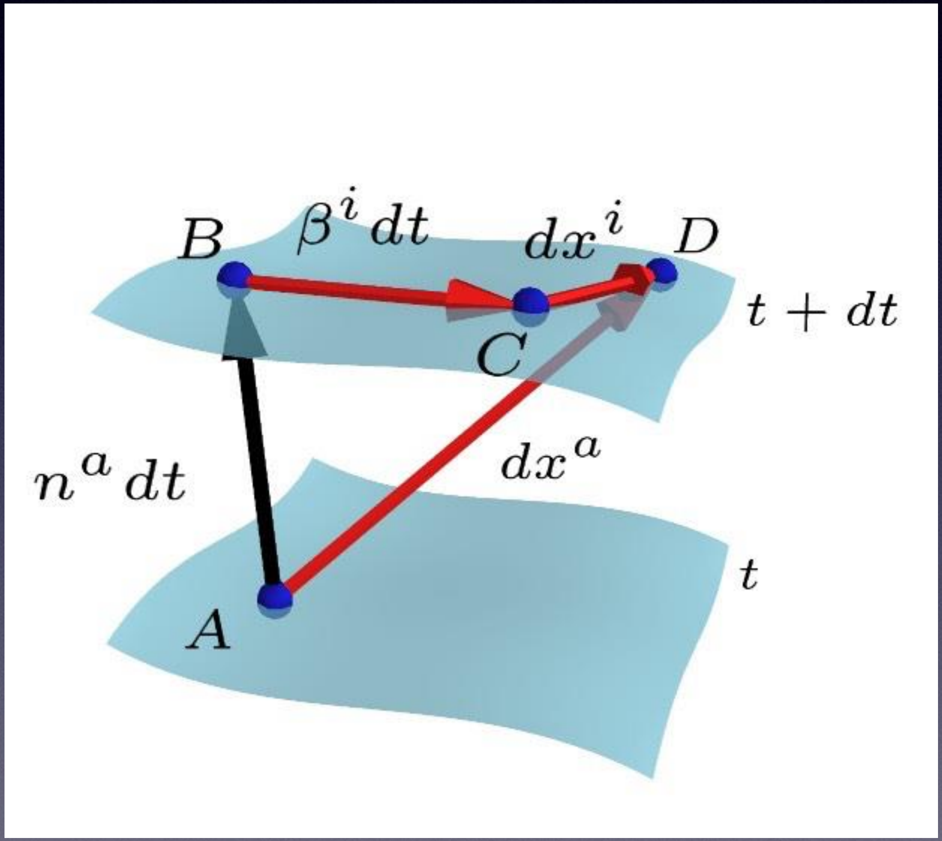
- Recall: *I had not expected that one could formulate the exact solution in such a simple way…*
- Can find exact solutions only under special circumstances
- In general need to employ approximations
- Most suitable for binary mergers: numerical relativity



Mark Scheel, SXS collaboration

3+1 Decomposition

- Cast Einstein's equations as Cauchy problem
- Introduce spatial foliation (slices of constant coordinate time t)
- Splits equations into
  - Constraint equations
  - Evolution equations

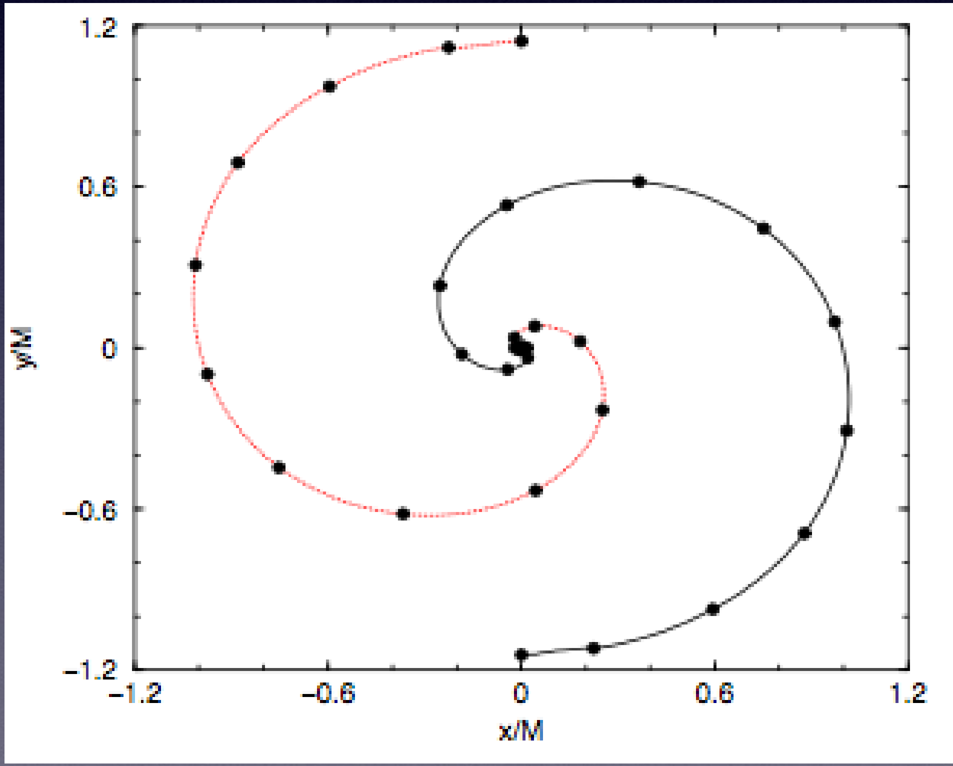


Decades of experimentation…

- Instabilities…
  - Formulation of evolution equations
- Coordinate freedom…
  - E.g.: how to choose slicing
- Curvature singularities…

...and a breakthrough!

- First successful simulations of binary black hole mergers in 2005
  - Pretorius
  - Campanelli *et.al.*
  - Baker *et.al.*



Campanelli *et.al., 2006*

Use coordinate freedom to avoid singularity…

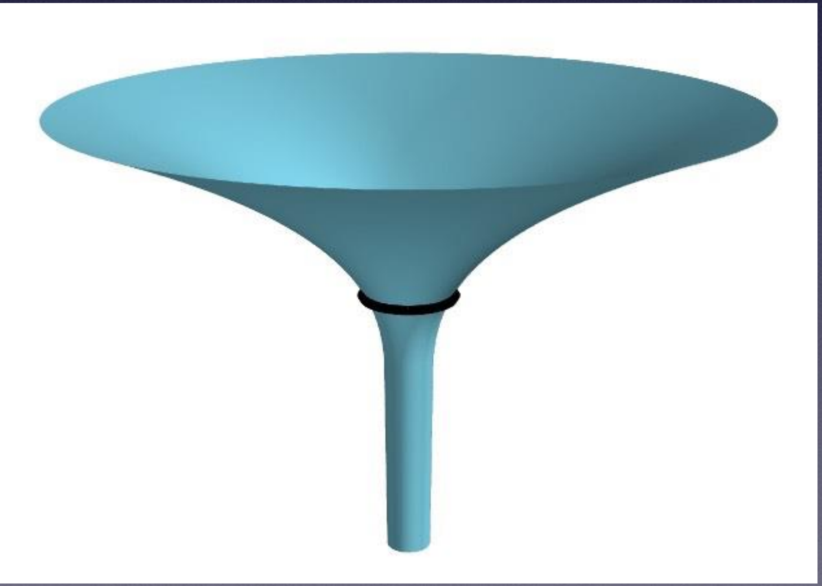
- Campanelli *et.al.* and Baker *et.al.* used special "slicing condition" that made simulations work almost miraculously
- Hannam *et.al.* (2007) tried same slicing condition for Schwarzschild black holes
  - Renders Schwarzschild spacetime in different coordinates
  - Spatial slices now feature "trumpet" geometries…

Trumpet geometries

$$ds^2 = -\frac{R-M}{R+M}dT^2 + \frac{2M}{R}dTdR + \left(1+\frac{M}{R}\right)^2 (dR^2 + R^2 d\Omega^2)$$

Dennison & TWB, 2014

- Construct embedding diagram as before
- Geometry now resembles trumpet
  - Does not reach curvature singularity
  - Perfect for numerical simulations



More on the geometry of black holes…

… Watch press conference on “groundbreaking results” from Event Horizon Telescope tomorrow, 9 am EST.

Image credit: University of Arizona

Newtonian gravity

- Gravitational force

$$F = G\frac{m_g M_\oplus}{R_\oplus^2} = m_g\, g$$

with $g = GM_\oplus / R_\oplus^2 = 9.81\text{m/s}^2$

- Second law

$$F = m_i\, a$$

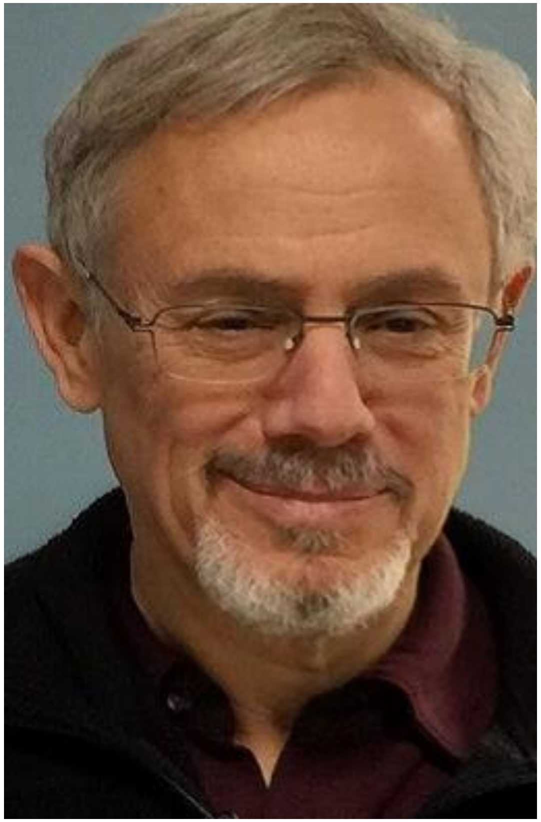
- Combine:

$$a = \frac{m_g}{m_i} g$$

MATHEMATICAL FRONTIERS
Mathematics of Gravitational Waves

Manuela Campanelli,
Rochester Institute of Technology

Thomas Baumgarte,
Bowdoin College



Mark Green,
UCLA (moderator)

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MATHEMATICAL FRONTIERS
2019 Monthly Webinar Series, 2-3pm ET

February 12: *Machine Learning for Materials Science**

March 12: *Mathematics of Privacy**

April 9: *Mathematics of Gravitational Waves*

May 14: *Algebraic Geometry*

June 11: *Mathematics of Transportation*

July 9: *Cryptography & Cybersecurity*

August 13: *Machine Learning in Medicine*

** Recordings posted*

September 10: *Logic and Foundations*

October 8: *Mathematics of Quantum Physics*

November 12: *Quantum Encryption*

December 10: *Machine Learning for Text*

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